EXTREME POINTS AND EXTREMUM PROBLEMS IN H_1

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The class H_1 consists of all functions f which are analytic in the open unit disc, and for which

$$||f|| = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})| d\theta$$

is finite. With this norm, H_1 is a Banach space, whose unit sphere will be denoted by S; that is, S is the set of all $f \in H_1$ with $||f|| \leqq 1$.

We are concerned in this paper with (a) the identification of the extreme points of S and some geometric properties of the set of these extreme points, (b) the closure of Pf (the set of all functions of the form $p \cdot f$, where p ranges over the polynomials and f is a fixed function in H_1) in various topologies, and (c) the structure of the set of those $f \in S$ which maximize a given bounded linear functional on H_1.

We find that the factorization $f = M_f Q_f$ (see Lemma 1.3), which was apparently first used by Beurling [1], is of basic importance in these problems.

Our results are summarized at the beginning of Sections II, III, and IV.

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I. PRELIMINARIES

1.1 Let C be the boundary of the open unit disc U in the complex plane. If $f \in H_1$, then $f(e^{i\theta})$, which we define to be $\lim_{r \to 1} f(re^{i\theta})$, exists almost everywhere on C and differs from 0 for almost all $e^{i\theta}$, unless f is identically 0. Moreover, the one-to-one correspondence between an $f \in H_1$ and its boundary function is an isometric embedding of H_1 in L_1, the Banach space of all Lebesgue integrable functions on C, normed by

$$(1.1.1) \qquad ||f|| = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta})| d\theta \ .$$

Thus (1.1.1) may be taken as the norm in H_1. We also have

$$(1.1.2) \qquad \lim_{r \to 1} \int_{-\pi}^{\pi} |f(re^{i\theta}) - f(e^{i\theta})| d\theta = 0$$

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