ABELIAN GROUPS CHARACTERIZED BY THEIR INDEPENDENT SUBSETS

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1. Introduction. Recently, the theory of abelian groups has become an active field. This note is devoted to it. One of the important theorems, called " the theorem on the subgroups of a free abelian group U_n of finite rank n [2, p. 145] will be studied here. From now on the term "group" will be used instead of "abelian group", for simplicity. Since the notations, the definitions, and the terminologies vary for different authors, we refer to Kurosh [2] and Kaplansky [1] as standards. For example, for the definitions of the height of an element x in a primary group G (denoted by $h_G(x)$), of the lowest layer of a primary group we refer to [2], and the definitions, of $Z(n)$, of $Z(p^\infty)$ we refer to [1]. Moreover, the subgroups spanned by the subset $\{u_\alpha\}$ of a given group G is denoted by $(\{u_\alpha\})$, and in a primary group we write $o(x) = n$ if the order of the element x is equal to p^n.

For convenience, the following terminology is adopted.

I. A group G has property (A), if for any non-zero element x of G there exists a cyclic direct summand of G containing x.

II. A group G has property (B), if G is a direct sum of cyclic groups, and for any subgroup H of G there exists a basis $\{h_\alpha\}$ of H and a basis $\{g_\beta\}$ of G such that for any $h_\alpha \in \{h_\alpha\}$ we can find a $g_\alpha \in \{g_\beta\}$ with the property $h_\alpha \in (g_\alpha)$.

III. A group G has property (C), if for any independent subset $\{h_\alpha\}$ of G there exists another independent subset $\{g_\alpha\}$ of G such that $h_\alpha \in (g_\alpha)$ and $(\{g_\alpha\})$ is a direct summand of G.

The purpose of this paper is to give an analysis of these classes of groups. In particular, we show that a free group U_n of finite rank $n \geqq 2$ has properties (A) and (B) but not (C).

GENERAL LEMMA. *A torsion group G has property* (A), (B) *or* (C) *respectively, if and only if each of its primary components has property* (A), (B) *or* (C) *respectively.*

Proof. We decompose G into its primary components,

$$G = \sum_p \oplus G_p \,.$$

We prove that G has property (C) if and only if each G_p has property (C). If for an independent subset $\{x_\alpha^{(p)}\} \subset G_p$, we can find an independent subset $\{g_\alpha\} \subset G$ such that $x_\alpha^{(p)} \in (g_\alpha)$ for all α, and the subgroup

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