CONJUGATE SERIES AND A THEOREM OF PALEY

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1. Introduction. It is known that a trigonometric series

$$\sum_{-\infty}^{\infty} a_n e^{inx} \tag{1}$$

does not have to satisfy condition on the size of its coefficients stronger than the trivial one

$$\sum_{-\infty}^{\infty} |a_n|^2 < \infty$$

in order to be the Fourier series of a continuous function. One theorem which gives precise content to this general statement is the following: *If* $\{w_n\}_{-\infty}^{\infty}$ *is a sequence of non-negative numbers such that*

$$\sum_{-\infty}^{\infty} |a_n| w_n < \infty$$

whenever (1) *is the Fourier series of a continuous function, then*

$$\sum_{-\infty}^{\infty} w_n^2 < \infty \ .$$

The fact that (1) is the Fourier series of a continuous function does not by any means imply the same for

$$\sum_{0}^{\infty} a_n e^{inx} \tag{2}$$

Therefore the following rather neglected theorem of Paley [5] lies deeper than the result just stated.

THEOREM 1 (Paley). *If* $\{w_n\}_0^{\infty}$ *is a sequence of non-negative numbers such that*

$$\sum_{0}^{\infty} |a_n| w_n < \infty \tag{3}$$

whenever (2) *is the Fourier series of a continuous function, then*

$$\sum_{0}^{\infty} w_n^2 < \infty \ . \tag{4}$$

In the next section we offer a new and simple proof of this theorem. The proof depends on the fact that the conjugate series of a Fourier-

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