PROOF OF THE FUNDAMENTAL THEOREM ON IMPLICIT FUNCTIONS BY USE OF COMPOSITE GRADIENT CORRECTIONS

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1. Introduction. Many methods have been employed for establishing the classical result, Theorem 2.1, concerning the existence of functions $x_i(t)$ satisfying a system

$$f_j(x\,;\,t) = 0, \qquad (j = 1, 2, \cdots, n) \tag{1.1}$$

of n equations in n unknowns $(x_1, \cdots, x_n) = x$ with $(t_1, \cdots, t_p) = t$, where all variables and functions are real valued, and $f_j(\alpha\,;\,\beta) = 0$. The object of this article is to present a new proof of the theorem by a constructive method of successive approximations involving corrections related to the gradients in x-space of the functions $f_j(x\,;\,\beta)$.

To establish Theorem 2.1, a sequence $x^{(m)}(t)$ with $x^{(0)}(t) = \alpha$ will be defined, where $x^{(m)}(t)$ is obtained by adding to $x^{(m-1)}(t)$ a vector correction $\Delta x^{(m-1)}(t)$ which is equal to a certain constant, ρ, times the vector sum of corrections parallel to the gradients of the $f_j(x\,;\,\beta)$ at $x = \alpha$. The vector $\Delta x^{(m-1)}(t)$, for a fixed t, is a special case of the corresponding correction of an iterative process for solving a general system $g_j(x) = 0$, $(j = 1, \cdots, k)$, $k \geqq n$, introduced by the authors in a previous article [2].

For a particular system (1.1), the method of the present paper would be applicable to obtaining values of the $x_i(t)$ by use of a digital computing machine for any t sufficiently near $t = \beta$. Section 6 in [2] describes a related small arc method with the same objective; the two methods differ in the values of the arguments used in fundamental matrices which appear with similar roles in [2] and below. The method of [2] might be superior computationally to the method of the present paper. However, in § 6 in [2], Theorem 2.1 below was employed as a starting point. Thus the present paper shows that the composite gradient method is effective to establish the supporting Theorem 2.1 as well as the related small arc method of [2] for computing values of the implicit functions.

In connection with the present article, it is pertinent to mention the proof of Theorem 2.1 by E. Goursat, [1], extended by William L. Hart, [3] and [4], to various infinite systems. In the Goursat method for (1.1), a system

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