A FINITE ALGORITHM FOR THE SOLUTION OF CONSISTENT LINEAR EQUATIONS AND INEQUALITIES AND FOR THE TCHEBYCHEFF APPROXIMATION OF INCONSISTENT LINEAR EQUATIONS

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I. Introduction. The three problems of the title are treated here from a unified geometric standpoint, and an algorithm is presented for their solution. Algorithms for these problems already exist in [8], [4], [5], and [6]. In the present algorithm, each problem is reinterpreted as one of finding the lowest points (if any exist) of a polytope in an Euclidean space; the techniques of steepest descent and elimination of variables are then combined to work downward from vertex to vertex. Professor T. S. Motzkin kindly called our attention to references [8] and [3] in which a similar viewpoint is exploited. Our thanks are also due to the referee for helpful suggestions, and to Norman Levine, who has coded the algorithm on the "704" automatic digital computer.

Section II which follows provides a description of the problems and an outline of the algorithm. Section III contains a detailed statement of the algorithm. Section IV concerns the special case of n variables and $n+1$ equations or inequalities. Section V is devoted to a proof of finiteness of the algorithm. In § VI, relationships among the three above problems and linear programming are discussed. In particular, it is shown that a simple modification of the algorithm will permit its application to linear programming.

II. The problems defined. Let $\{A^1, \cdots, A^m\}$ be a subset of E_n with $m \geqq n$, and let b denote a fixed point of E_m. We emphasize that throughout the paper the matrix A composed of the rows $A^1, \cdots, A^m$ is assumed to be of rank n. The notation (u, v) will be used for $\sum u_i v_i$, and a vector $\bar{x} = (x_1, \cdots, x_{n+1})$ will be said to have *abscissa* $x = (x_1, \cdots, x_n)$ and *ordinate* x_{n+1}. The word *polytope* denotes the intersection of a finite number of half-spaces. Define the polytope

$$\mathscr{P} = \left\{ \bar{x} : x_{n+1} \geqq \max_{1 \leqq i \leqq m} [(A^i, x) - b_i] \right\} .$$

If $\mathscr{P}$ intersects the half-space $\{\bar{x} : x_{n+1} \leqq 0\}$, then the system

Received August 1, 1957, and in revised form March 11, 1958. Parts of this paper were presented to the American Mathematical Society as follows: June 1958 abstract 589, August 1957 abstract 748. The paper was presented by title to the Conference on Matrix Computation, Wayne State University, September 1957.