LINEAR INEQUALITIES AND QUADRATIC FORMS

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1. Introduction. There are known criteria for a quadratic form to be positive definite, and criteria for a system of linear inequalities to have a solution. In this paper the two problems are shown to be related. The principal theorem is Theorem 5.1.

2. Definitions and Notation. We will consider a quadratic form

$$Z(x) \equiv \sum_1^n a_{ij}x_ix_j, \text{ with } a_{ij} = a_{ji} ,$$

and ask whether it is positive in the first orthant, i.e., whether it is positive for non-negative values of the x_i.

If $Z(x) > 0$ for $x \geqq 0$, we call it *conditionally definite* and if $Z(x) \geqq$ for $x \geqq 0$, we call it *conditionally semi-definite*. (Since we will only be concerned with positive definiteness, we will omit the word "positive" throughout the paper.) Finally, if $Z(x) \geqq 0$ when $x \geqq 0$ and $Z(x) > 0$ when $x > 0$, we call $Z(x)$ *conditionally almost-definite*.

As a matter of notation, we recall that $Ax \geq 0$ or $x \geq 0$ means that at least one component of the vector in question is positive.

In discussing $Z(x)$ we shall have occasion to refer to the form obtained by setting $x_{k_1}, x_{k_2}, \cdots, x_{k_s}$ equal to zero, that is, the form

$$\sum_{i,j \neq k_1, \cdots, k_s} a_{ij}x_ix_{ij} .$$

We shall call this a principal minor of $Z(x)$ and denote it $Z_{k_1 \cdots k_s}(x)$. In referring to the corresponding matrix, $A^{k_4 \cdots k_s}$ we will assume x has the appropriate number of components when we write $A^{k_1 \cdots k_s}x$.

3. Quadratic forms in the first orthant. We first prove a theorem which is not strictly necessary but may be some intrinsic interest. It concerns the game whose matrix is $A = (a_{ij})$ and whose value is v. (For completeness we remind the reader of the following definition of the value v of a game with matrix $B = (b_{ij})$, $i = 1, \cdots, m$; $j = 1, \cdots, n$. Let X be the set of vectors $x = (x_1, \cdots, x_m)$ with $x_1 \geqq 0$ and $\sum_1^m x_i = 1$; Y the set of $y = (y_1, \cdots, y_n)$ with $y_j \geqq 0$ and $\sum_1^n y_j = 1$. Then it can be shown that

$$\max_{x \in X} \min_{y \in Y} \sum b_{ij}x_iy_j = \min_{y \in Y} \max_{x \in X} \sum b_{ij}x_ix_j ,$$

and this quantity is called the value of the game with matrix B).

Received December 9, 1957.