ON A COMMUTATIVE EXTENSION OF A COMMUTATIVE BANACH ALGEBRA

CIPRIAN FOIAS

Let A be a commutative Banach algebra without identity such that

(1.a) there exists an approximate identity (i.e. there exists a net $\{u_\alpha\} \subset A$, so that $\| u_\alpha \| = 1$ and $u_\alpha x \to x$ for all $x \in A$);

(1.b) if $\hat{A}$ designates Gelfand's representation of A [3], and M the space of regular maximal ideals of A, then the boundary of M with respect to $\hat{A}$, is equal to M[1].

Let $\mathscr{L}(A)$ be the algebra of all bounded linear operators on A; the mapping $x \to T_x$ of A into $\mathscr{L}(A)$, where $T_x y = xy$, $y \in A$, is isomorphic and isometric (by (1.a)) onto a subalgebra $\tilde{A}$ of $\mathscr{L}(A)$,

Let $\mathscr{A}$ be the set of those operators $T \in \mathscr{L}(A)$ which commute with all $T_x \in \tilde{A}$, that is such that

$$T(xy) = (Tx)y = x(Ty), \qquad x, y \in A. \tag{1}$$

LEMMA (i). *For all $T \in \mathscr{A}$, we have $T = \lim T_{Tu_\alpha}$, the limit being considered in the strong operator topology.*

(ii) *$\mathscr{A}$ is the closure of $\tilde{A}$ in the strong operator topology.*

(iii) *$\mathscr{A}$ is the largest commutative subalgebra of $\mathscr{L}(A)$ which contains $\tilde{A}$.*

(iv) *$\tilde{A}$ is an ideal in $\mathscr{A}$.*

Proof. From (1) and (1.a), it follows that

$$T_{Tu_\alpha} y = Tu_\alpha \cdot y = T(u_\alpha y) \to Ty$$

for all $T \in \mathscr{A}$ and $y \in A$, hence (i) is proved. (ii) results from (i). Concerning (iii), it is enough to prove that $\mathscr{A}$ is commutative; or, by (i) and (1)

$$T_1 T_2 x = \lim T_{T_1 u_\alpha} T_2 x = T_2 \lim T_{T_1 u_\alpha} x = T_2 T_1 x, \qquad T_1, T_2 \in \mathscr{A}, \ x \in A.$$

If $T \in \mathscr{A}$ and $x, y \in A$, then

$$TT_x y = T(xy) = (Tx)y = T_{Tx} y,$$

hence

Received March 21, 1958.

[1] For example this condition is satisfied if $\mathscr{A}$ is regular or selfadjoint, see [**3**, p. 81].