TWO NON-SEPARABLE COMPLETE METRIC SPACES DEFINED ON [0, 1]

BURNETT MEYER AND H D SPRINKLE

Let $\mathfrak{M}$ be the set of all Lebesgue measurable subsets of the closed interval [0, 1], and let $A, B \in \mathfrak{M}$. It is well-known that $\mathfrak{M}$ becomes a pseudo-metric space if distance is defined by

$$d(A, B) = m(A - B) + m(B - A) = m[(A - B) \cup (B - A)] ,$$

m denoting the Lebesgue measure. See [1, pp. 31–32]. It is the purpose of this paper to extend $\mathfrak{M}$ to include the non-measurable sets and to examine some of the properties of the resulting space.

If we remove the restriction that A and B be measurable, and let them be any subsets of [0, 1], then if

$$\rho(A, B) = m^*(A - B) + m^*(B - A), \text{ and } \delta(A, B) = m^*[A - B) \cup (B - A)]$$

(where m^* denotes the exterior Lebesgue measure), it is easily seen that pseudo-metric spaces $\mathfrak{S}$ and $\mathfrak{T}$ are obtained, corresponding to ρ and δ respectively. The properties which we discuss of $\mathfrak{S}$ and $\mathfrak{T}$ are the same and are proved analogously, so we shall state and prove our results for the space $\mathfrak{S}$ only, it being understood that similar theorems and proofs hold for $\mathfrak{T}$.

LEMMA 1. *A necessary and sufficient condition that* $\rho(A, B) = 0$ *is the existence of sets* Z_1 *and* Z_2, *both of Lebesgue measure zero, such that* $A \cup Z_1 = B \cup Z_2$.

Necessity. If $\rho(A, B) = 0$, then $m(A - B) = m(B - A) = 0$. Since $A \cup (B - A) = A \cup B = B \cup A = B \cup (A - B)$, Z_1 and Z_2 may be taken as $B - A$ and $A - B$, respectively.

Sufficiency. If $A \cup Z_1 = B \cup Z_2$, then

$$\rho(A, B) \leqq \rho(A, A \cup Z_1) + \rho(A \cup Z_1, B \cup Z_2) + \rho(B \cup Z_2, B) = 0$$

The relation $\rho(A, B) = 0$ is seen to be an equivalence relation defined on the elements of $\mathfrak{S}$; hence, those elements are partitioned into equivalence classes. Let $[A]$ denote the equivalence class which contains A. It is clear that if $C \in [A]$ and $D \in [B]$, then $\rho(A, B) = \rho(C, D)$. If $\mathfrak{S}^*$ is the set of all equivalence classes defined above, and if $\rho([A], [B]) = \rho(A, B)$, then $\mathfrak{S}^*$ becomes a metric space with the metric $\rho([A], [B])$.

Received June 11, 1958.