ON THE NUMBER OF BI-COLORED GRAPHS

FRANK HARARY

1. Introduction. This is an extension of papers [2, 3, 4] whose notation and terminology will be used. The main result is a formulation of the generating function or counting polynomial of bicolored graphs, obtained by the enumeration methods of Pólya [6]. A modification of the method yields the number of balanced signed graphs, solving a problem proposed in [5]. In the process of enumerating bicolored graphs, we consider two binary operations on permutation groups called "cartesian product" and "exponentiation" which are abstractly but not permutationally equivalent to the direct product and Pólya's "Gruppenkranz" [6], respectively.

A *graph* consists of a finite set of *points* together with a prescribed subset of the collection of all *lines*, i.e., unordered pairs of distinct points. Two points are *adjacent* if there is a line joining them. A graph is *k-chromatic*[1] if each of the points can be assigned one of k given colors so that any two adjacent points have different colors. A graph is k-colored if it is k-chromatic and its points are colored so that all k colors are used. More precisely, a k-colored graph is a pair (G, f) where G is a graph and f is a function from the set of points of G *onto* the set of numbers $1, 2, \cdots, k$ such that if a and b are adjacent points, then $f(a) \neq f(b)$. Two graphs are *isomorphic* if there exists a one-to-one adjacency preserving transformation between their sets of points. Two k-colored graphs are *chromaticallg isomorphic* if there is a color preserving isomorphism between them. Thus (G_1, f_1) is chromatically isomorphic with (G_2, f_2) if there is an isomorphism $\theta\colon G_1 \rightarrow G_2$ and a permutation $\omega\colon \{1, \cdots, k\} \rightarrow \{1, \cdots, k\}$ such that $\omega(f_1(a)) = f_2(\theta(a))$ for every point a in G_1. Let $g_{p,q}^{(k)}$ be the number of chromatically nonisomorphic k-colored graphs with p points and q lines, and let the corresponding generating function be

$$g_p^{(k)}(x) = \sum_{q=0}^{p(p-1)/2} g_{p,q}^{(k)} x^q . \tag{1}$$

We first derive the number of bicolored graphs, $k = 2$, and then discuss the formula for $k = 3$. The problem remains open for $k > 2$.

Received November 4, 1957, and in revised forms March 28, 1958, and June 16, 1958. This work was supported by a grant from the National Science Foundation. The author is deeply grateful to the referee for making several insightful comments of clarification.

[1] This definition is different from that of Dirac [1]. According to Dirac, a graph has *chromatic number k* if it is k-chromatic but not $(k-1)$-chromatic as defined here.