A NOTE ON POLYNOMIAL AND SEPARABLE GAMES

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1. Introduction. A two-person zero-sum game Γ is called polynomial-like or *separable* if its payoff function is of the form

$$M(x, y) = \sum_{i=1}^{n} f_i(x) g_i(y),$$

where x and y are elements of any strategy sets X and Y. Important special cases of separable games are those in which X and Y are bounded (usually compact) subsets of Euclidean spaces and M is a polynomial in the coordinates of x and y. These latter are called *polynomial games*.

It is a basic and fairly elementary fact concerning separable games [1], that, if optimal strategies exist, then these can always be chosen to be finite mixed strategies. We consider here the inverse question: Given a pair of finite mixed strategies, does there exist a separable (respectively, polynomial) game whose unique optimal strategies are the given pair ? In case either X or Y is *finite* the answer is known to be in the negative. We here show, however, that.

THEOREM 1. *If X and Y are metric spaces containing infinitely many points and μ and ν are any finite mixed strategies on X and Y respectively, then there is a payoff M, bounded continuous and separable on $X \times Y$, such that the associated game has μ and ν as unique optimal strategies.*

COROLLARY. *If X is a metric space containing infinitely many points and μ is any finite mixed strategy on X, then there is a skew-symmetric payoff M, bounded continuous and separable on $X \times X$ such that the associated symmetric game has μ as the unique optimal strategy.*

For the case of polynomial games we show:

THEOREM 2. *If X and Y are bounded subsets of Euclidean spaces whose closures contain infinitely many cluster points, then for any finite mixed strategies μ and ν there exists a polynomial payoff function M such that the associated game has μ and ν as its unique optimal strategies.*

(An analogous corollary holds here, also.)

Concerning Theorem 2, we remark that Glicksberg and Gross, [2],

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