MINIMAL COVERINGS OF PAIRS BY TRIPLES

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1. Introduction. Let F be a finite set with n members, $n \geqq 3$. An *F-covering of pairs by triples*, which we abbreviate F-copt, is a set S of triples of distinct members of F which has the property that each pair of distinct members of F is contained in at least one member of S. If n is a positive integer, $n \geqq 3$, then an n-copt is an F-copt for the set $F = \{1, 2, \cdots, n\}$. We assume throughout that $n \geqq 3$.

For any finite set A, let $C(A)$ denote the number of members of A. An F-copt S is *minimal* if $C(S) \leqq C(S')$ for every F-copt S'. If $n \equiv 1 \pmod 6$ or $n \equiv 3 \pmod 6$, then a minimal n-copt S turns out to be *exact* in the sense that each pair is contained in exactly one member of S. Such exact coverings are called *Steiner triple systems*. The existence of Steiner triple systems for all n (of form $6h + 1$ or $6h + 3$) was proved by M. Reiss [2] in 1859.

Let S be a minimal n-copt and let $C(S) = \mu(n)$. The main result of this paper is obtained in §2, where we determine $\mu(n)$ explicitly for $n \geqq 3$. In § 3 we discuss certain properties of minimal n-copts, and give several methods for constructing minimal n-copts.

2. Determination of $\mu(n)$. Let S be a minimal n-copt. For each integer i, $1 \leqq i \leqq n$, we define $\alpha(i)$ to be the number of members of S that contain i. Then

$$\sum_{i=1}^{n} \alpha(i) = 3 \cdot C(S) .$$

Since i must appear in members of S with $n - 1$ other numbers we have $\alpha(i) \geqq [n/2]$. ($[x]$ is the largest integer which is not greater than x.) Thus,

$$\mu(n) = C(S) \geqq \frac{n}{3}\left[\frac{n}{2}\right] . \tag{1}$$

Since $(n/3)\,[n/2]$ may not be an integer, we define $\varphi(n)$ to be the least integer which is not less than $(n/3)\,[n/2]$. It is easy to compute

$$\varphi(n) = \begin{cases} n^2/6 \text{ if } n = 6k , \\ n(n-1)/6 \text{ if } n = 6k+1 \text{ or } n = 6k+3 , \\ (n^2+2)/6 \text{ if } n = 6k+2 \text{ or } n = 6k+4 , \\ (n^2-n+4)/6 \text{ if } n = 6k+5. \end{cases} \tag{2}$$

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