MODULES WHOSE ANNIHILATORS ARE DIRECT SUMMANDS

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Introduction. Let B be a ring with an identity element, and let M be a right B-module. The set of all elements b in B such that $Mb = (0)$ is called the annihilator of M, and will be denoted by $(0 : M)$. It is a natural question to ask under what circumstances the ideal $(0 : M)$ is a direct summand of B. If B is a semi-simple ring with minimum condition, for example, then every ideal is a direct summand, and there is no problem. We shall be concerned with a ring B, not assumed to be semi-simple, which is a crossed product $\Delta(G, H, \rho)$ of a finite group G and a division ring Δ, with factor set ρ. In particular, B may be the group algebra of a finite group with coefficients in a field. The purpose of this note is to obtain necessary and sufficient conditions on the structure of the module M in order that its annihilator $(0 : M)$ be a direct summand of B.

Our interest in the problem stems chiefly from the fact that the the modules whose annihilators are direct summands turn out to be precisely the modules for which the pairing defined in § 2 of [1] is regular in the sense of [1, p. 281]. The main results of [1], given in § 5 and § 6, are based upon the assumption that the pairing is regular, and establish a connection between the structure of the module M relative to the set of B-endomorphisms of M and the structure of a certain ideal in B, called the nucleus of M, which is the uniquely determined complementary ideal to $(0 : M)$ when $(0 : M)$ is a direct summand.

2. Familiarity with crossed products and their connection with projective representations of finite groups is assumed (see [1, § 2]). In this section we recall some of the properties of a crossed product, and introduce, in a more general, and at the same time, much simpler fashion, the pairing defined in a special case by formula (7) of [1]. Let $G = \{1, s, t, \cdots\}$ be a finite group, Δ a division ring and $B = \Delta(G, H, \rho)$ a crossed product of G and Δ with correspondence $s \to \bar{s} = s^H$ from G to the group of automorphisms of Δ, and factor set $\{\rho_{s,t}\}$. There exist elements $\{b_1, b_s, \cdots\}$ in B in one-to-one correspondence with the elements of G, such that every element of B can be expressed uniquely in the form $\sum b_s \xi_s$, with coefficients ξ_s in Δ. The multiplication in B is determined by the equations

$$(1) \qquad b_s b_t = b_{st} \rho_{s,t} \,; \quad \xi b_s = b_s \xi^{\bar{s}}, \qquad \xi \in \Delta \,.$$

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