SUBDIRECT SUMS AND INFINITE ABELIAN GROUPS

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1. Definitions. Let G be a group, and suppose G is a subgroup of the direct sum $\sum_{a\in I}\oplus H_a$ of the collection of groups $\{H_a\}_{a\in I}$. If the projection of G into H_a is onto H_a for each $a \in I$, then G is said to be a *subdirect sum* of the groups $\{H_a\}_{a\in I}$. (Only weak direct and subdirect sums are considered here.) If a group G is isomorphic to a subdirect sum of the groups $\{H_a\}_{a\in I}$, then G is said to be *represented* as a subdirect sum of the groups $\{H_a\}_{a\in I}$. A group is called a *rational group* if it is a subgroup of a $Z(p^\infty)$ group or a subgroup of the additive group of rational numbers.

2. THEOREM. *Every Abelian group can be represented as a subdirect sum of rational groups where the subdirect sum intersects each of the rational groups non-trivially.*

Proof. G is isomorphic to a subgroup of some divisible group, and thus can be represented as a subdirect sum G' of rational group $\{H_a\}_{a\in I}$. Let $(h_1, h_2, \cdots, h_a, \cdots)$ be an element of G'. Let $(h_1, h_2, \cdots, h_a, \cdots)\beta_1 = (k_1, h_2, \cdots, h_a, \cdots)$, where $k_1 = h_1$ if $G' \cap H_1 \neq 0$, and $k_1 = 0$ if $G' \cap H_1 = 0$. Assume β_c has been defined for $c < b$. Define

$$(h_1, h_2, \cdots, h_a, \cdots)\beta_b = (k_1, k_2, \cdots, k_b, h_{b+1}, \cdots)$$

where $k_b = h_b$ if $H_b \cap (\bigcup_{c<b} G'\beta_c) \neq 0$, and $k_b = 0$ otherwise. Each β_a preserves addition because each is a projection. Let $(h_1, h_2, \cdots, h_a, \cdots) \neq (0, 0, \cdots, 0, \cdots)$ and let

$$(h_1, h_2, \cdots, h_a, \cdots)\beta_a = (k_1, k_2, \cdots, k_a, h_{a+1}, h_{a+2}, \cdots)\,.$$

Only a finite number of the coordinates of $(h_1, h_2, \cdots, h_a, \cdots)$ are not 0. Let them be $h_{a_1}, h_{a_2}, \cdots, h_{a_n}$, where $a_1 < a_2 < \cdots < a_n$. If $a < a_n$, then

$$\begin{aligned}&(h_1, h_2, \cdots, h_a, \cdots)\beta_a\\ &\quad = (k_1, k_2, \cdots, k_a, h_{a+1}, \cdots, h_{a_n}, h_{a_n+1}, \cdots) \neq (0, 0, \cdots, 0, \cdots)\end{aligned}$$

since $h_{a_n} \neq 0$. Assume $a \geqq a_n$. If $n=1$ and $a_1=1$, then $(h_1, h_2, \cdots, h_a, \cdots) = (h_{a_1}, 0, 0, \cdots, 0, \cdots) \in G'$ and $G' \cap H_1 \neq 0$ so that $(h_{a_1}, 0, 0, \cdots$ [illegible] $= (h_{a_1}, 0, 0, \cdots, 0, \cdots)$. That is, $k_{a_1} = h_{a_1} \neq 0$, and hence $(h_1, h_2, \cdots$ [illegible] $\neq (0, 0, \cdots, 0, \cdots)$. If $n = 1$ and $a_n \neq 1$, then $(0, 0, \cdots, h_{a_1}, 0, 0, \cdots) \in G'$ and also in $G'\beta_c$ for all $c < a_1$. Thus $H_{a_1} \cap (\bigcup_{c<a_1} G'\beta_c) \neq 0$, and

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