ON INTEGRATION OF 1-FORMS

MAURICE SION

1. Introduction. It has been noted by several people that in order to define the integral of some differential 1-form ω along a curve C, the latter need not be of bounded variation. For example, in the extreme (and trivial) case where ω is the differential of some function f, the integral can be defined as the difference of the values assumed by f at the end-points of C. No condition on C is necessary. H. Whithney [4], with J. H. Wolfe, by the introduction of certain norms, has found general abstract spaces of curves along which the integral of 1-forms satisfying certain conditions can be defined. In fact, H. Whitney considers integration of p-forms with $p \geq 1$. In a previous paper [2], we obtained rather awkward conditions for a decent integral to exist that depended on the number of higher derivatives of ω on C.

In this paper, we consider 1-forms ω possessing 'higher derivatives' on C in a sense somewhat different from that due to H. Whitney [3] which we used previously. A Lipschitz type condition on the remainders of the Taylor expansion is imposed (see 4.1.). We define the α-variation of a curve as the supremum of sums of αth powers of chords (see 2.7) and show that the integral of ω along C exists if the α-variation of C is bounded, where α is related to the number of 'higher derivatives' of ω on C. Under somewhat stronger hypotheses on C, we show that this integral is an anti-derivative of ω on C.

2. Notation and basic definitions. Throughout this paper, N is a positive integer and we use the following notation.

2.1. E denotes Euclidean $(N+1)$-space.

2.2. $\|x\| = \left(\sum_{i=0}^{N} x_i^2\right)^{1/2}$ for $x \in E$.

2.3. diam $U = \sup\{d : d = \|x - y\| \text{ for some } x \in U \text{ and } y \in U\}$

2.4. φ is a continuous function on the closed unit enterval to E and $C = \text{range } \varphi$.

2.5. $\mathscr{S}$ is the set of all subdivisions of the unit interval, i.e. functions T on $\{0, 1, \cdots, k\}$ for some positive integer k such that:
$T(0) = 0, \quad T(k) = 1, \quad T(i-1) < T(i)$ for $i = 1, \cdots, k$

2.6. $[T/a, b] = \{i : a \leq T(i-1) < T(i) \leq b\}$

2.7. $V_\alpha(a, b) = \sup_{T \in \mathscr{S}} \sum_{i \in [T/a,b]} \|\varphi(T(i-1) - \varphi(T(i))\|^\alpha$

Received August 25, 1958.