ON A THEOREM DUE TO SZ.-NAGY

R. S. PHILLIPS

B. Sz.-Nagy [4] has proved the following theorem:

THEOREM A. *Let* $[T_t;\ t \geqq 0]$ *be a strongly continuous semi-group of contraction operators on a Hilbert space* H. *Then there exists a group of unitary operators* $[\mathbf{U}_t,\ -\infty < t < \infty]$ *on a larger Hilbert space* $\mathbf{H}$ *such that*

$$(1) \qquad T_t y = \mathbf{P}\mathbf{U}_t y, \qquad y \in H,\ t \geqq 0\ ;$$

here $\mathbf{P}$ *is the projection operator with range* H. *Then space* $\mathbf{H}$ *can be chosen in a minimal fashion so that* $[\mathbf{U}_t H;\ -\infty < t < \infty]$ *spans* $\mathbf{H}$. *In this case* $[\mathbf{U}_t]$ *is strongly continuous and the structure* $\{\mathbf{H}, \mathbf{U}_t, H\}$ *is determined to within an isomorphism.*[1]

The infinitesimal generator L of the semi-group $[T_t]$ is defined by

$$(2) \qquad \lim_{\delta \to 0+} \delta^{-1}[T_\delta y - y] = Ly$$

for all $y \in H$ for which this limit exists. The operator L is linear and closed with dense domain, $\mathfrak{D}(L)$ (see [1]). It is shown in [2] that L is maximal dissipative in the sense that

$$(3) \qquad (y, Ly) + (Ly, y) \leqq 0, \qquad y \in \mathfrak{D}(L)\ ,$$

and L being maximal with respect to this property. Since $[\mathbf{U}_t]$ is a semi-group as well as a group of operators, the infinitesimal generator $\mathbf{L}$ of $[\mathbf{U}_t]$ also shares these properties; however in the case of a group of unitary operators $i\mathbf{L}$ is in addition self-adjoint.

The purpose of this note is to study the relation between L and $\mathbf{L}$. It turns out that L is a restriction of $\mathbf{L}$ only when L is maximal symmetric. In general L is neither a restriction nor a projection of $\mathbf{L}$; in fact $\mathfrak{D}(\mathbf{L}) \cap H$ may contain only the zero element. Nevertheless we shall obtain $\mathbf{H}$, $\mathbf{L}$, and $[\mathbf{U}_t]$ directly from L, our principal tool being the discrete analogue of the above theorem, which is also due to Sz.-Nagy [4], namely

THEOREM B. *Let* J *be a contraction operator on a Hilbert space* H. *Then there exists a unitary operator* $\mathbf{J}$ *on a larger Hilbert space* $\mathbf{H}$ *such that*

$$(4) \qquad J^n y = \mathbf{P}\mathbf{J}^n y, \qquad y \in H,\ n \geqq 0;$$

here $\mathbf{P}$ *is the projection operator with range* H. *The space* $\mathbf{H}$ *can be*

Received September 24, 1958. This paper was written under the sponsorship of the National Science Foundation, contract NSF G-4231.

[1] Two structures $\{\mathbf{H}, \mathbf{U}_t, H\}$ and $\{\mathbf{H}', \mathbf{U}'_t, H\}$ are isomorphic if there is a unitary map $\mathbf{V}$ of $\mathbf{H}$ onto $\mathbf{H}'$ which is the identity on H and is such that $\mathbf{V}\mathbf{U}_t y = \mathbf{U}'_t\mathbf{V}y$ for all $y \in \mathbf{H}$.