

PROPERTIES PRESERVED IN SUBDIRECT PRODUCTS

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1. Introduction. A characterization is obtained of those sentences S of the predicate calculus such that S holds for a subdirect product of general algebraic or relational systems¹ whenever it holds for each component system. We consider formulas in a first order language equipped with symbols for the operations and relations of the systems under consideration, and, in particular, with a symbol for the identity relation. An *atomic formula* is one obtained by inserting terms in the argument places of a relation symbol. A *positive formula* is one that can be built up from atomic formulas by means of conjunction, disjunction, and of universal and existential quantification (but without using negation). A *special Horn formula* is one of the form $P \supset F$ where P is a positive formula and F is an atomic formula, or any formula obtained from such formulas by conjunction and universal quantification. A *sentence* is a formula without free variables. As a corollary to our main theorem we obtain the following:

A sentence has the property that it holds for a subdirect product of systems whenever it holds for each component system if and only if it is equivalent to a special Horn sentence.

An example of a special Horn sentence is provided by the condition for an associative ring to be semisimple in the sense of Jacobson [7, Proposition 1, p. 9], which is expressed by the following sentence:

$$\forall z \cdot [\forall x \forall y \cdot xzy + u = xzyu \wedge uxzy = xzyu] \supset z = 0 .$$

We admit among subdirect products the subdirect product of an empty set of systems, which, from the definition, proves to be a trivial system with a single element and all relations universal. The sole effect of excluding this trivial case would be to admit in special Horn sentences clauses $\sim P$ along with the clauses $P \supset F$.

A. Horn [6] considered the more general class of all sentences obtained by universal and existential quantification from conjunctions of formulas of the type $P \supset F$ (or $\sim P$), where P is a conjunction of atomic formulas and F an atomic formula. Horn showed that all such sentences are preserved under (full) direct products, while C. C. Chang and Anne C. Morel [4] showed that there are sentences preserved under direct product that are not equivalent to any such Horn sentence. The problem of characterizing syntactically those sentences preserved under direct

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¹ This concept is due to Tarski; see [13], [14].