AN INTERPOLATION THEOREM IN THE PREDICATE CALCULUS

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1. Introduction. In studying the formal structure of sentences whose validity is preserved under passage from an algebraic system to a homomorphic image of the system, we have had occasion to use a lemma from formal logic. A proof of this lemma, our Interpolation Theorem, can be given within the theory of deductive inference, as formalized by Gentzen. Gentzen's theory is rather complicated and perhaps not generally well known. Moreover, the use of any formalized system of deductive logic seems to an extent alien to the primarily algebraic nature of our intended application. Therefore we give here a proof of the Interpolation Theorem that lies entirely within the theory of models: our arguments are as far as possible in the spirit of abstract algebra, and, in particular, borrow nothing from formal logic beyond an understanding of the intended meaning, herein precisely defined, of the conventional symbolism.

The Interpolation Theorem deals with sentences of the Predicate Calculus. Roughly, these are sentences that can be build up using the usual logical connectives, symbols denoting operations (or functions), symbols denoting relations (or predicates), and variables whose range is individual elements of the systems under consideration, but no variables ranging over operations, relations, or sets. The theorem takes the same form whether or not we admit a predicate denoting identity, with suitable axioms, to the predicate calculus. For technical reasons we admit as sentential connectives only the signs for negation, conjunction and disjunction (regarding "if ··· then" as a defined concept), together with signs 0 and 1 for truth and falsehood. For each occurrence of a relation symbol in a sentence S, there is a unique maximal chain of well formed formulas, all containing the given occurrence and each occurring as a proper part of the next. The given occurrence of the relation symbol will be called *positive* if the number of formulas in this chain that begin with the negation sign is even, and *negative* if this number is odd. If S is in prenex disjunctive form, this criterion takes the simpler form that an occurrence is negative if and only if it is preceded by the negation sign.

INTERPOLATION THEOREM, *Let S and T be sentences such that S implies T. Then there exists a sentence M such that S implies M and M*

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1. See [**5**] and [**9**], Chapter XV.