ON THE RADIUS OF UNIVALENCE OF THE FUNCTION $\exp z^2 \int_0^z \exp(-t^2)dt$

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1. Introduction. We shall determine the radius of univalence ρ_u of the function

$$E(z) = e^{z^2}\int_0^z e^{-t^2}dt \ . \tag{1.1}$$

We shall write $E(z) = w = u(x, y) + iv(x, y)$. On the imaginary axis we have $u = 0$ and v, regarded as a function of y, has a single maximum at the solution $y = \rho$ of

$$2yv(0, y) = 1 \ .$$

The value of ρ to eight decimal places has been determined by Lash Miller and Gordon [1] and is

$$\rho = 0.92413887 \ . \tag{1.2}$$

It is evident that $\rho_u \leqq \rho$. We shall prove the following theorem.

THEOREM. *The number ρ is the radius of univalence of $E(z)$.*

Recently, the radius of univalence of the error function

$$erf(z) = \int_0^z e^{-t^2}dt$$

was determined [2]. It is interesting to note that when proceeding from $erf(z)$ to $E(z)$ we meet an entirely different situation. In the case of $erf(z)$, points z_1, z_2 closest to the origin and such that $erf(z_1)=erf(z_2)$ are conjugate complex and lie far apart from each other. In the case of $E(z)$ points of that nature can be found in an arbitrarily small neigborhood of the point $z = i\rho$.

The actual situation is made clear by the diagram and tables given below. In Fig. 1 we show the curves $R = |E| =$ constant and $\gamma = arg\, E =$ constant in the square $0 \leqq x \leqq 1.5$, $0 \leqq y \leqq 1.5$ of the z-plane. The table shows the values of E for z on the curve C (defined below). The values given were obtained by summing an adequate number of terms of the power series on the Datatron 205 at the California Institute of Technology; some were checked by comparison with the tables of Karpov [4, 5] from which values of $E(z)$ can be obtained.

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