ON ONE-TO-ONE HARMONIC MAPPINGS

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In this paper we shall prove the following:

THEOREM. *Let* $z = z(w)$ $(z = x + iy,\ w = u + iv)$ *be a one-to-one harmonic mapping of the disc* $|w| < 1$ *onto the disc* $|z| < 1$ *such that* $z(0) = 0$. *Then we have for* $|w| < 1$ *the estimate*

$$(1) \qquad |z_u|^2 + |z_v|^2 \geqq \frac{2}{\pi^2} .$$

As an improvement of an earlier result established in [1] J. C. C. Nitsche [4] showed that under the above conditions the inequality

$$(2) \qquad (|z_u|^2 + |z_v|^2)_{w=0} \geqq \frac{1}{2}$$

is satisfied[1]. In contrast to (2) the estimate (1) holds throughout the unit disc $|w| < 1$, but the constant involved is smaller than that of Nitsche.

In order to establish (1) we shall make use of a known result on harmonic functions (the analogue of the Schwarz Lemma)[2]. For the sake of completeness the proof of it will be given here.

LEMMA. *Let* $z = z(w) = x(w) + iy(w)$ *be a complex-valued harmonic function in the disc* $|w| < 1$. *Furthermore, let* $z(0) = 0$ *and* $|z(w)| < 1$ *for* $|w| < 1$. *Then we have the inequality*

$$(3) \qquad |z(w)| \leqq \frac{4}{\pi} \arctan |w| \qquad |w| < 1.$$

Proof. Let θ be an arbitrary real number, and $f(w)$ be the function, which is regular-analytic in the disc $|w| < 1$ and satisfies the relations $f(0) = 0$ and

$$(4) \qquad \Re f(w) = x(w) \cos \theta + y(w) \sin \theta .$$

On account of our hypotheses we have

$$(5) \qquad |\Re f(w)| < 1 \qquad |w| < 1,$$

hence,

[1] For further references see [**2**].

[2] See Polya-Szegö [**5**], p. 140.

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