ON THE VAN KAMPEN THEOREM

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1. Introduction. The van Kampen theorem provides a defining set of generators and relations for the fundamental group of the union of two topological spaces X and Y where the fundamental groups of X, Y, and their intersection are given by defining sets of generators and relations. An intrinsic, purely group-theoretic formulation has been given by Fox using his direct limits of systems of groups [4]; however, the corresponding abstract proof had not been worked out. The present paper supplies such a proof (distilled from an earlier proof by Fox of the van Kampen theorem) to a natural generalization of the van Kampen theorem, which includes for example, in addition to the original theorem, the determination of the fundamental group of the union of an increasing nest of open sets each of whose groups is known [2].

In proving the principal result, Theorem (3.1), we depart from the usual development of the fundamental group in that paths and loops are not required to have the fixed unit interval as domains. In particular, a *path* a is a continuous mapping of the interval $[0, \|a\|]$ into the space in question for some $\|a\| \geq 0$. For paths $a : [0, \|a\|] \rightarrow X$ and $b : [0, \|b\|] \rightarrow X$ which satisfy $a(\|a\|) = b(0)$, we define the *product path* $a \cdot b$ by

$$a \cdot b(t) = \begin{cases} a(t) \text{ for } 0 \leq t \leq \|a\| \\ b(t - \|a\|) \text{ for } \|a\| \leq t \leq \|a\| + \|b\| . \end{cases}$$

Thus, path multiplication is associative. Paths a and b, having the same initial and terminal points, are *equivalent*, denoted by $a \simeq b$, iff there exists a collection of paths h_s: $[0, \|h_s\|] \rightarrow X$, $0 \leq s \leq 1$, such that

$h_0 = a$ and $h_1 = b$,
$h_s(0) = a(0) = b(0)$,
$h_s(\|h_s\|) = a(\|a\|) = b(\|b\|)$,
$\|h_s\|$ is a continuous function of s,
$h_s(t)$ is simultaneously continuous in s and t.

We note that, for any path a and positive number t, there is a path b equivalent to a with $\|b\| = t$. Furthermore, $\|h_s\|$ can always be taken as a linear function of s and thus, in view of the preceding sentence, may be arranged to be constant. The induced multiplication of equivalence classes of paths and the definitions of the fundamental groupoid and group of X are made in the usual way.

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