NON-ABELIAN ORDERED GROUPS

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1. Introduction. In this paper we prove some theorems about non-abelian o-groups, and give some methods of constructing such groups. Most of the literature on o-groups is concerned with abelian o-groups, and the examples in print of non-abelian o-groups are few. Iwasawa [8] proves that any free group can be ordered, and he also gives some additional examples of o-groups. Vinogradov [15] shows that the free product of two o-groups A and B can be ordered so as to preserve the given orders. Chehata [1] gives an example of an o-group that is simple. [3] and [11] contain examples of o-groups. Most of the theorems in this paper give methods for constructing o-groups. For example, in § 3 we study the o-automorphisms of an o-group G. For every group A of o-automorphisms of G that can be ordered we can construct a new o-group H that contains A and G. H is the natural splitting extension of G by A. In § 5 the relationship between central extensions and bilinear mappings is exploited. It is shown that any skew-symmetric real matrix can be used to construct o-groups. In § 6 some o-groups of rank 2 are constructed. In § 4 a study is made of the ordered extensions of a subgroup of the reals. One of the main results is a necessary and sufficient condition for such an extension to split. The principal tool used throughout is the extension theory of Schreier [14].

2. Notation and Terminology. The notation of [3] is used throughout. In particular, the notation and results from § 2 [3, pp. 517–518] are used repeatedly. Unless otherwise stated the group operation will always be addition and 0 will denote a group identity. N and N' are o-groups with elements $a, b, c, \cdots$ and $a', b', c', \cdots$ respectively. G is a normal o-extension of N by N'. We identify G with its representation $G' = N' \times N$, where

$$(a', a) + (b', b) = (a' + b', f(a', b') + ar(b') + b)$$

and (a', a) is positive if $a' > 0$ or $a' = 0$ and $a > 0$. See [3] for the properties of the factor mapping f and the representative function r.

θ will always denote a trivial homomorphism of a group onto the identity element of some other group. For an o-group H, let $A(H)$ be the group of all o-automorphisms of H. For an abelian o-group K, let $D(K)$ be the d-closure or completion of K. In particular, $D(K)$ is a vector space over the rationals and there is a natural extension of the order

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