p-CYCLIC MATRICES: A GENERALIZATION OF THE YOUNG-FRANKEL SUCCESSIVE OVERRELAXATION SCHEME

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1. Introduction. The Young-Frankel [11, 5] successive overrelaxation scheme, which has been shown [11, pp. 104–109] to be applicable to the numerical solution of partial equations of elliptic type, can be described as follows. If the system of linear equations to be solved is

$$M\vec{x} = \vec{k}\,, \tag{1}$$

where the $n \times n$ matrix $M = (m_{i,j})$ is such that $m_{i,i} \neq 0$ for $i = 1, 2, \cdots, n$, then the iterative sequence, defined by the successive overrelaxation scheme, is given by

$$x_i^{(l+1)} = \omega\left\{\sum_{j=1}^{i-1} b_{i,j}x_j^{(l+1)} + \sum_{j=i+1}^{n} b_{i,j}x_j^{(l)} + c_i\right\} + (1-\omega)x_i^{(l)} \tag{2}$$

where $x_i^{(0)}$ is arbitrary, $i = 1, 2, \cdots, n$, and where

$$b_{i,j} = \begin{cases} -m_{i,j}/m_{i,i}, & i \neq j \\ 0, & i = j \end{cases}, \tag{3}$$

and

$$c_i = k_i/m_{i,i}, \qquad i = 1, 2, \cdots, n\,. \tag{4}$$

With certain assumptions, Young [11] has shown that, for suitable choice of the *relaxation factor* ω, relatively rapid convergence of the iterative process (2) is assured. These hypotheses are satisfied by the usual five-point difference approximation to $-\nabla \cdot (k \nabla u) = S$, $k > 0$, in the plane [11, 9].

We show that successive overrelaxation can be considered as a special case of a more general iterative scheme applicable to the wider class of *p-cyclic* matrices, to be defined below. Indeed, ordinary successive (point) overrelaxation, as well its generalization [1] to successive block (line) overrelaxation, is just the special case $p = 2$ of the iterative scheme we shall now define.

2. p-cyclic matrices. We begin with the following

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