SPACES WHOSE FINEST UNIFORMITY IS METRIC

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Those metrizable spaces for which the finest uniform structure is induced by a metric have attracted a certain amount of attention, and M. Atsuji [1] has collected and extended a list of characterizations of them, regarded as uniform spaces. J. Nagata [6] and B. Levshenko [4] have given topological characterizations of these spaces. This note extends Atsuji's list and gives an analogous list of topological characterizations.

I am indebted to the referee for assistance with the references and improvements in the proofs.

Recall that a metric space (or a subset of a metric space) is said to be *uniformly discrete* if for some $\varepsilon > 0$, the distance between two different points is always at least ε.

THEOREM 1. *For a metric uniform space S, either of the following properties implies that the metric uniformity is the finest compatible with the topology; thus they are equivalent to the properties* (1)-(8) *of* [1, Theorem 1].

(9) *All bounded continuous real-valued functions are uniformly continuous.*

(10) *Every closed discrete subspace of S is uniformly discrete.*

THEOREM 2. *For a metrizable topological space S, the following properties are mutually equivalent:*

(a) *The finest uniformity on S is a metric uniformity.*

(b) *The set of all non-isolated points of S is compact.*

(c) *Every subset of S has a compact boundary.*

(d) *Every closed set has a compact boundary.*

(e) *Every closed continuous image of S is metrizable.*

(f) *Every Hausdorff quotient space of S is metrizable.*

(g) *Every Hausdorff quotient space satisfies the first axiom of countability.*

(h) *Every closed set in S has a countable basis of neighborhoods.*

The equivalence of (a) and (b) in Theorem 2 is due to Nagata [6]. Levshenko has given three conditions equivalent to (b) [4]. One is that S is a regular space having a countable family of locally finite converings such that every locally finite covering has a refinement in this family; the other two are obtained by replacing "locally finite" in both

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