SOME ITERATIVE METHODS FOR DETERMINING ZEROS OF FUNCTIONS OF A COMPLEX VARIABLE

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1. Introduction. Innumerable schemes have been developed for finding approximate numerical values for the zeros of a function $f(z)$. It is remarkable, considering the diversity of motivating ideas behind these methods, that so many of them (including the classical Newton-Raphson method) are essentially nothing more than combinations of Bernoulli's method and, in effect, the origin shifting procedure of Horner. In turn these combinations may be formulated in terms of the power series expansion of a fraction $g(z)/f(z) = \sum_{n=0}^{\infty} c_n z^n$. We present here several methods of utilizing the information implicit in the coefficients c_n, obtaining several new iterative schemes for approximating zeros and reformulating some well known ones—frequently in such a way as to provide a simpler method of computation.

We assume that for computational purposes, if $g(z) = \sum_{n=0}^{\infty} a_n z^n$ and $f(z) = \sum_{n=0}^{\infty} b_n z^n$, then the coefficients for $g(z)/f(z) = \sum_{n=0}^{\infty} c_n z^n$ may conveniently be obtained recursively by the standard equations

$$
\begin{aligned}
&c_0 b_0 = a_0 \\
(1)\qquad &c_0 b_1 + c_1 b_0 = a_1 \\
&\cdots\cdots\cdots \\
&c_0 b_n + c_1 b_{n-1} + \cdots + c_n b_0 = a_n
\end{aligned}
$$

where for convenience, we make $b_0 = 1$ so that each coefficient is given as a product sum $c_n = a_{n-1} - b_1 c_{n-1} - \cdots - b_n c_0$.

By the order of an iteration we mean the concept introduced by Schröder [19]. If an iteration produces a sequence $z_n \to \alpha$, it is of order N if

$$\frac{|z_{n+1} - \alpha|}{|z_n - \alpha|^N} \to c \neq 0 ,$$

c a constant.

2. Basic lemma. We base our results on a lemma which specifies the contribution to the coefficients c_n of those zeros closest to the origin. Let $f(z)$ and $g(z)$ be analytic in $|z| < R$ and let $f(z)$ have zeros α_i, $i =$

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