# ON STRICTLY SEMI-SIMPLE BANACH ALGEBRAS

EDITH H. LUCHINS

**I. Introduction.** Define the *strict radical* of an algebra to be the intersection of just those of its two-sided ideals which are regular maximal right ideals. Call the algebra *strictly semi-simple* (sss) if its strict radical is the zero ideal. This note proves that the strict radical of a real Banach algebra $B$ contains the set of topologically nilpotent elements of $B$. Also, it gives a condition which is both necessary and sufficient for $B$ to be sss.

**II. Preliminaries.** For any ring or algebra $A$ let $T(A)$ denote the set of all those two-sided ideals in $A$ which are regular maximal right ideals. The intersection of the elements of $T(A)$ is the *strict radical* of $A$. $A$ is *strictly semi-simple* (sss) if its strict radical is the zero ideal.

LEMMA 1. *Let $I$ be a two-sided ideal in the algebra (ring) $A$. Then the following are equivalent:*

(a) *$I \in T(A)$, that is, $I$ is a regular maximal right ideal.*

(b) *$I$ is a regular maximal left ideal.*

(c) *$A/I$ is a division algebra (division ring).*

*Proof.* Use is made of the theorem [4, Theorem 24.6.1] that a division algebra has no proper right or left ideals and that an algebra with no proper right ideals either is trivial or is a division algebra.

If (a) holds, then $A/I$ has no proper right ideals. Now $A/I$ is not trivial since if $j$ is a left unit element of $A$ modulo $I$, $j' \cdot j' = j' \neq 0$ (where $x'$ denotes the image of $x \in A$ under the canonical homomorphism of $A$ onto $A/I$). The cited theorem shows $A/I$ is a division algebra. Thus (a) implies (c) and, similarly, (b) implies (c). Moreover, if (a) holds, then $j'$ is a left identity for $A/I$ and hence an identity for it, so that $I$ is regular with $j$ as its associated unit element. If $I \subset L$, $L$ a left ideal in $A$, then $L/I$ is a left ideal in $A/I$, and an improper ideal by the cited theorem, so that $L = I$ or $A$ and $I$ is a regular maximal left ideal. Thus (a) implies (b).

Suppose (c) holds and $e'$ is a unit of $A/I$. Then $I$ is regular with

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