CONJUGATE SERIES IN SEVERAL VARIABLES

HENRY HELSON

1. Introduction. For any trigonometric series

$$\sum_{-\infty}^{\infty} a_n e^{inx}$$

the conjugate series is defined to be

$$\sum_{-\infty}^{\infty} -ia_n \, sg \, n \, e^{inx} ,$$

with the convention that $sg\,0 = 0$. A series in two variables

$$(1) \qquad \sum_{m,n=-\infty}^{\infty} a_{mn} e^{i(mx+ny)}$$

can be conjugated with respect to x, with respect to y, or with respect to both variables at once. The last possibility gives the series

$$(2) \qquad \sum_{m,n=-\infty}^{\infty} -a_{mn} \, sg(mn) \, e^{i(mx+ny)} ,$$

and this has been called [3, 5] the conjugate of (1).

For series in one variable, there are several theorems which state that a trigonometric series belonging to some function class always has conjugate in the same, or perhaps in a different function class. These theorems lead to similar results for series in several variables conjugated with respect to *one* of the variables. For example, one proves very easily that

$$(3) \qquad \int |\tilde{f}^x| d\sigma \leqq A \int |f| \log^+ |f| d\sigma + B ,$$

where $\tilde{f}^x$ is the function conjugate to f with respect to x, $d\sigma = d\sigma(x, y)$ is invariant measure on the torus, and A, B are absolute constants.

In conjugation with respect to x, the coefficients a_{mn} of (1) are multiplied by $-i$ in the right half-plane and by i in the left half-plane. Conjugation in y involves the upper and lower half-planes in the same way. Any half-plane bounded by a line of rational slope can be transformed by a linear change of variables into, say, the upper half-plane, and so there is a whole family of notions of conjugacy with corresponding theorems. If, however, we divide the plane by a line of irrational slope, it is not so clear how to prove the same theorems, although the definition of conjugacy with respect to any line is at hand. It is even

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