ORDERINGS OF THE SUCCESSIVE OVERRELAXATION SCHEME

RICHARD S. VARGA

1. Introduction. One of the more frequently used iterative methods [11, 14, 18] in numerically solving self-adjoint partial difference equations of elliptic type:

$$(1) \qquad \sum_{j=1}^{n} a_{i,j}x_j = k_i, \qquad a_{i,i} \neq 0, \qquad 1 \leqq i \leqq n \,,$$

is the Young-Frankel *successive overrelaxation scheme* [16, 4]. If superscripts denote the iteration indices, then the successive overrelaxation scheme is defined by

$$(2) \qquad x_i^{(n+1)} = \omega\left\{\sum_{j=1}^{i-1} b_{i,j}x_j^{(n+1)} + \sum_{j=i+1}^{n} b_{i,j}x_j^{(n)} + g_i\right\} + (1-\omega)x_i^{(n)} \,,$$

where

$$(2') \qquad b_{i,j} = \left\{\begin{matrix} -a_{i,j}/a_{i,i}, & i \neq j \\ 0 \,, & i = j \end{matrix}\right\}; \; g_i = k_i/a_{i,i}, \qquad 1 \leqq i, j \leqq n \,.$$

The parameter ω is the *relaxation factor.*

Since the introduction of this method, there has remained the question of the effect of different orderings of the equations of (1) on the rate of convergence of the overrelaxation scheme. Young [16] introduced the concept of a *consistent ordering* of the unknowns for a class of matrices satisfying his definition of *property* (A), and he conjectured [17] that, with certain additional assumptions, these consistent orderings were optimal[1] in the sense that, among all orderings, the consistent orderings give the fastest convergent iterative scheme for the case of $\omega = 1$ of (2).

The problem of the relationship between orderings and rates of convergence has been recently investigated by Heller [6], whose approach was combinatorial. Assuming the $n \times n$ matrix $A \equiv \|a_{i,j}\|$ of (1) to be multi-diagonal, Heller concentrated on the problem of finding all orderings whose associated Gauss-Seidel iterative method, the special case of (2) with $\omega = 1$, had the same eigenvalues as the eigenvalues of the Gauss-Seidel method based on the "usual ordering."

Our approach to the question of orderings is based on the Perron-

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[1] For some preliminary results on this conjecture for optimum orderings, see [**17**].