ON TCHEBYCHEFF POLYNOMIALS

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1. Introduction. Let C be a closed bounded set having an infinite number of points. There is a unique polynomial $T_n(z)$ of degree n, and with one as coefficient of z^n, such that if $P_n(z)$ is any other polynomial with the same normalization,

$$M_n = \max_{z \in C} |T_n(z)| < \max_{z \in C} |P_n(z)| . \tag{1.1}$$

This is the Tchebycheff polynomial of degree n associated with C.

1.1. Assume that C has positive capacity, used throughout to mean logarithmic capacity, and a connected complement D. The conductor potential for such C is a real valued function $U(z)$ defined in D with the properties: (1.2) $U(z)$ is harmonic at finite points of D, (1.3) $U(z) - \log|z|$ is regular at infinity and zero there, (1.4) there is a number $\rho > -\infty$ such that $U(z) > \rho$ for z in D, (1.5) if $\{z_i\}$ is a convergent sequence of points with limit point on the boundary of D, then $\lim U(z_i) = \rho$, except perhaps when the limit point belongs to a subset of the boundary of capacity zero. The function $U(z)$ has a unique representation as a Lebesgue-Stieltjes integral

$$U(z) = \int \log|z - t| d\mu . \tag{1.6}$$

where μ is a completely additive, positive set function defined for Borel measurable sets, if it is specified that the carrier of μ consist of boundary points of D. [2].

1.2. Fejér [1] proved that the zeros of $T_n(z)$ lie in the convex hull H of C. The consequence

$$|z_{ni}| \leqq R , \tag{1.7}$$

where z_{ni} is a zero of $T_n(z)$, and R is a finite constant independent of n, will be sufficient for later reference. Let

$$\rho_n = \frac{1}{n} \log M_n . \tag{1.8}$$

Szegö [3] proved that

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