ON THE SIMILARITY TRANSFORMATION BETWEEN A MATRIX AND ITS TRANPOSE

OLGA TAUSKY AND HANS ZASSENHAUS

It was observed by one of the authors that a matrix transforming a companion matrix into its transpose is symmetric. The following two questions arise:

I. Does there exist for every square matrix with coefficients in a field a non-singular symmetric matrix transforming it into its transpose?
II. Under which conditions is every matrix transforming a square matrix into its transpose symmetric?

The answer is provided by

**THEOREM 1.** For every $n \times n$ matrix $A = (a_{ik})$ with coefficients in a field $F$ there is a non-singular symmetric matrix transforming $A$ into its transpose $A^\tau$.

**THEOREM 2.** Every non-singular matrix transforming $A$ into its transpose is symmetric if and only if the minimal polynomial of $A$ is equal to its characteristic polynomial i.e. if $A$ is similar to a companion matrix.

**Proof.** Let $T = (t_{ik})$ be a solution matrix of the system $\Sigma(A)$ of the linear homogeneous equations.

\[
\begin{align*}
(1) & \quad TA - A^\tau T = 0 \\
(2) & \quad T - T^\tau = 0.
\end{align*}
\]

The system $\Sigma(A)$ is equivalent to the system

\[
\begin{align*}
(3) & \quad TA - A^\tau T^\tau = 0 \\
(4) & \quad T - T^\tau = 0
\end{align*}
\]

which states that $T$ and $TA$ are symmetric. This system involves $n^2 - n$ equations and hence is of rank $n^2 - n$ at most. Thus there are at least $n$ linearly independent solutions of $\Sigma(A)$.

On the other hand it is well known that there is a non-singular matrix $T_0$ satisfying

\[T_0 A T_0^{-1} = A^\tau,
\]

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This part of the proof was provided by the referee. Our own argument was more lengthy.