ON THE SIMILARITY TRANSFORMATION BETWEEN A MATRIX AND ITS TRANSPOSE

OLGA TAUSSKY AND HANS ZASSENHAUS

It was observed by one of the authors that a matrix transforming a companion matrix into its transpose is symmetric. The following two questions arise:

I. Does there exist for every square matrix with coefficients in a field a non-singular symmetric matrix transforming it into its transpose?

II. Under which conditions is every matrix transforming a square matrix into its transpose symmetric?

The answer is provided by

THEOREM 1. *For every $n \times n$ matrix $A = (\alpha_{ik})$ with coefficients in a field F there is a non-singular symmetric matrix transforming A into its transpose A^T.*

THEOREM 2. *Every non-singular matrix transforming A into its transpose is symmetric if and only if the minimal polynomial of A is equal to its characteristic polynomial* i.e. *if A is similar to a companion matrix.*

Proof. Let $T = (t_{ik})$ be a solution matrix of the system $\sum(A)$ of the linear homogeneous equations.

$$TA - A^T T = 0 \tag{1}$$

$$T - T^T = 0\ . \tag{2}$$

The system $\sum(A)$ is equivalent to the system

$$TA - A^T T^T = 0 \tag{3}$$

$$T - T^T = 0 \tag{4}$$

which states that T and TA are symmetric. This system involves $n^2 - n$ equations and hence is of rank $n^2 - n$ at most. Thus there are at least n linearly independent solutions of $\sum(A)$.[1]

On the other hand it is well known that there is a non-singular matrix T_0 satisfying

$$T_0 A T_0^{-1} = A^T\ ,$$

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[1] This part of the proof was provided by the referee. Our own argument was more lengthy.