# VARIATIONS ON A THEME OF CHEVALLEY

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**1. Introduction.** In this paper we use the methods of C. Chevalley to construct some simple groups and to gain for them the structural theorems of [3]. Among the groups obtained there are two new families of finite simple groups[1], not to be found in the list of E. Artin [1]. Whether the infinite groups constructed are new has not been settled yet.

Section 5 contains statements of the main results of [3]. In §§ 2, 3, 4 and 7, we define analogues of certain real forms of the Lie groups of type $A_l$, $D_l$ and $E_6$ (in the usual notation), and extend to them the structural properties of the groups of Chevalley. Sections 6 and 9 treat some identifications, and § 8 deals with the question of simplicity. In §§ 10 and 11, using the extra symmetry inherent in a Lie algebra of type $D_4$, we consider two modifications of the first construction which are, perhaps, of more interest since they produce groups which have no analogue in the classical complex-real case: in fact, a basic ingredient of each of these variants is a field automorphism of order 3. In Sections 12 and 13, it is proved that new finite simple groups are obtained[1], and their orders are given. Section 14 deals with an application to the theory of group representations, and § 15 with some concluding observations.

The notation is cumulative. We denote by $|S|$ the cardinality of the set $S$, by $K^*$ the multiplicative group of the field $K$, and by $C$ the complex field. An introduction to the standard Lie algebra terminology together with statements of the principal results in the classical theory can be found in [3, p. 15–19]. (Proofs are available in [8] or [10]).

**2. Roots and reflections.** We first introduce some notations. Relative to a Cartan decomposition of a simple complex Lie algebra of rank $l$, let $E$ be the real space generated by the roots, made into an Euclidean space in the usual way, and normalized as in [3, p. 17–18]. Relative to an ordering $\prec$ of the additive group generated by the roots, let $\Pi$ be the set of positive roots, and $a(1), a(2), \cdots, a(l)$ the fundamental roots. For each root $r = \Sigma z_i\, a(i)$, set $\Sigma z_i = ht\ r$, the *height* of $r$. The ordering $\prec$ can always be chosen so that $ht\ r < ht\ s$ implies $r \prec s$ (see [3, p. 20, $l$. 35–40]); suppose this is done. Assume now the existence of an automorphism $\sigma$ of $E$ of order 2 such that $\sigma\Pi = \Pi$. This restricts the type of algebra to $A_l$, $D_l$ $(l \geqq 4)$ or $E_6$ (see [3, p. 18]), and hence

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[1] Since the preparation of this paper, the author has learned that these groups have also been discovered by D. Hertzig [6], who has shown that they complete the list of finite simple algebraic groups.