SOME CONNECTIONS BETWEEN CONTINUED FRACTIONS AND CONVEX SETS

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The purpose of this paper is to develop certain connections between the continued fraction solutions and the convex set solutions to some of the moment problems. In particular, we shall develop some relations between the work of Wall [3], [4] on continued fractions and the work of Karlin and Shapley [1] on convex sets. The paper is divided into two parts:

I. Stieltjes-type continued fractions and convex sets.

II. Jacobi-type continued fractions and convex sets.

Two characterizations of the moment problem for the interval (0, 1), one by Riesz [2] in terms of convex closures and one in term of Hankel forms, are well known. The work of Karlin and Shapley [1] shows the equivalence of these two characterizations. A third characterization in terms of a Stieltjes-type continued fraction has been given by Wall [3], [4]. In part I we give an interpretation of the parameters in this continued fraction in terms of "distances" in certain convex bodies. This interpretation, through the work of Karlin and Shapley, immediately shows the equivalence of all three characterizations.

Solutions of the moment problem for the interval $(-1, 1)$, in terms of the Riesz condition and Hankel forms, are also well known. In part II we give a third solution in terms of a Jacobi-type continued fraction. Again, through an interpretation of the parameters in this continued fraction in terms of "distances" in certain convex bodies and an extension of the work of Karlin and Shapley, the equivalence of the three characterizations is immediate.

I. STIELTJES-TYPE CONTINUED FRACTIONS AND CONVEX SETS

1. The monotone Hausdorff moment problem. A sequence of real numbers $\{c_n\}(n = 0, 1, 2, \cdots)$ is called a monotone Hausdorff moment sequence if there exists a monotone nondecreasing real function $\phi(u)$, $0 \leqq u \leqq 1$, such that

$$c_n = \int_0^1 u^n d\phi(u), \qquad n = 0, 1, 2, \cdots .$$

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