ON TOEPLITZ MATRICES, ABSOLUTE CONTINUITY, AND UNITARY EQUIVALENCE

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1. Preliminaries. For $n = 0, \pm 1, \pm 2, \cdots$, let $\{c_n\}$ be real numbers satisfying

$$(1) \qquad c_0 = 0,\ c_{-n} = c_n \text{ and } \sum_1^\infty c_n^2 < \infty\ ,$$

and consider the associated real-valued, even function $f(\theta)$ of period 2 and of class $L^2[0, \pi]$ defined by

$$(2) \qquad f(\theta) \sim \sum_{-\infty}^{\infty} c_n e^{in\theta} = 2\sum_1^\infty c_n \cos n\theta\ .$$

(Throughout this paper it will be assumed for the sake of convenience that $c_0 = 0$. If $c_0 \neq 0$, T (see below) is modified merely by the addition of a multiple of the unit matrix.) Let $A = (a_{ij})$, where $a_{ij} = c_{i-j} (= c_{j-i})$ or $a_{ij} = 0$ according as $i < j$ or $i \geq j (i, j = 1, 2, \cdots)$, and define the Toeplitz matrix T and the Hankel matrices H and K by

$$(3) \qquad T = (c_{i-j}) = A + A^*,\ H = (c_{i+j-1}) \text{ and } K = (c_{i+j})\ .$$

The matrices T, H and K are real and Hermitian (symmetric).

Let J denote the matrix belonging to the quadratic form $2\sum_1^\infty x_n x_{n+1}$. The differential of its spectral matrix is given by $d\rho_{ij}(\theta) = 2\pi^{-1}\sin i\theta \sin j\theta d\theta$ (cf. Hilbert [5], p. 155, Hellinger [8], pp. 148 ff.). A direct calculation (cf. [11], Appendix 2) shows that

$$(4) \qquad T = F + K\ ,$$

where T and K are defined by (3), and F is given by

$$(5) \qquad F = \left(\int_0^\pi f(\theta) d\rho_{ij}(\theta)\right)\ ,$$

with $f(\theta)$ defined by (2) and (1). In particular, if $c_1 = 1$ and $c_n = 0$ for $n > 1$, then $f(\theta) = 2 \cos\theta$ and (5) is the spectral resolution of J (with the usual parameter λ being given by $\lambda = 2 \cos\theta$).

It should be noted that the L^2 assumption on the sequence $\{c_n\}$ in (1) does not imply the boundedness of the various matrices considered above, although of course, the existence, in the mean, of the integrals in (5) is assured. Moreover, all two factor products of the type A^2, AA^*, etc. surely exist and it can be verified that

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