ON THE IMBEDDABILITY OF THE REAL PROJECTIVE SPACES IN EUCLIDEAN SPACE[1]

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1. Introduction. Let P_n denote n-dimensional real projective space. This paper is concerned with the following question: What is the lowest dimensional Euclidean space in which P_n can be imbedded topologically or differentiably? Among previous results along this line, we may mention the following;

(a) If n is even, then P_n is a non-orientable manifold, and hence cannot be imbedded topologically in $(n+1)$-dimensional Euclidean space, R^{n+1}.

(b) For any integer $n > 1$, P_n cannot be imbedded topologically in R^{n+1}, because its mod 2 cohomology algebra, $H^*(P_n, Z_2)$, does not satisfy a certain condition given by R. Thom (see [6], Theorem V, 15).

(c) If $2^{k-1} \leqq n < 2^k$ then P_n cannot be imbedded topologically in Euclidean space of dimension $2^k - 1$. This result follows from knowledge of the Stiefel-Whitney classes of P_n (see Thom, loc. cit., Theorem III. 16 and E. Stiefel, [5]; also [4]).

In the present paper, we prove the following result: If $m = 2^k, k > 0$, then P_{3m-1} cannot be imbedded differentiably in R^{4m}. For example P_5 cannot be imbedded differentiably in R^8, nor can P_{11} be imbedded in R^{16}. Of course if $n > m$, P_n cannot, *a fortiori*, be imbedded differentiably in R^{4m}. Thus for many values of n our theorem is an improvement over previous results on this subject.[1]

The proof of this theorem depends on certain general results on the cohomology mod 2 of sphere bundles. These general results are formulated in § 2, and in § 3 the proof of the theorem is given. Finally in § 4 some open problems are discussed.

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2. Steenrod squares in a sphere bundle with vanishing characteristic. Let $p: E \to B$ be a locally trivial fibre space (in the sense of

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[1] This result partially solves a problem proposed by S. S. Chern (see Ann. Math., **60** (1954), p. 222). It follows that P_n cannot be imbedded in R^{n+2} for $n > 3$ except possibly in case $n = 2^k - 1, k > 2$. The case $n = 2^k - 1$ is still open. The importance of this problem, and some of its implications, were emphasized by H. Hopf in his address at the International Congress of Mathematicians held in Cambridge, Massachusetts in 1950. This address is published in volume I of the Proceedings of the Congress (see pp. 193-202).