CLASSES OF MINIMAL AND REPRESENTATIVE DOMAINS AND THEIR KERNEL FUNCTIONS

MICHAEL MASCHLER

1. Introduction. In connection with the problem of obtaining classes of conformally equivalent domains in the space of one or several complex variables, S. Bergman [3] introduced two kinds of canonical domains named *minimal domains* and *representative domains*. Since the mapping functions onto these domains were expressed in a closed form by using the Bergman kernel function and its derivatives, it was possible to deduce interesting properties of the kernel function which, in turn, provided more information about the canonical domains. (See S. Bergman [1] [3], M. Schiffer [9], M. Maschler [7]).

The object of this paper is to discuss "minimal domains" and "representative domains" with respect to certain subclasses of analytic functions, and to deduce solutions to some extremal problems. In addition, differential equations are obtained for the kernel function, which are valid for various classes of domains. The methods we use apply to the theory of functions of several complex variables as well, but first, the case of one complex variable should be clarified.

Let D be a plane domain having a boundary of positive capacity. We consider the class of analytic functions $w = f(z)$ which have single-valued, regular derivatives in D, and which possess developments of the form

$$w = (z - t) + a_{m+1}(z - t)^{m+1} + a_{m+2}(z - t)^{m+2} + \cdots \tag{1.1}$$

in the neighborhood of a point t in D. There exists one function in this class which maps D onto a domain having the smallest area[1]. This latter domain will be called an *m-minimal domain* with the origin as center. For $m = 1$ we obtain the ordinary minimal domains.

As $w = f(z)$ may be multivalued and non-univalent, one has to extend the theory of the kernel function to domains on a Riemann surface, which may have "identified points", (That is, points which correspond to a single point of a univalent domain, under a conformal mapping). The ideas of this extension are not new and are treated here for the

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[1] The area is defined by $\iint_D |f'(z)|^2 \, d\omega$, where $d\omega$ is the area element.