ON RADICALS AND CONTINUITY OF HOMOMORPHISMS INTO BANACH ALGEBRAS

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1. Introduction. All Banach algebras considered are over the real field and all homomorphisms considered are algebraic (real-linear). An algebra is called semi-simple, strongly semi-simple, or strictly semi-simple, if its Jacobson radical [5], Segal radical [10], or strict radical [8], respectively, is the zero ideal; that is, if its regular maximal right ideals, its regular maximal two-sided ideals, or those of its two-sided ideals which are regular maximal right ideals, intersect in the zero ideal. Rickart [9, Corollary 6.3] proved that a semi-simple commutative Banach algebra has the property that every homomorphism of a Banach algebra into it is continuous. Call an algebra with this property an *absolute* algebra. Yood [12, Theorem 3.5] proved that every homomorphism of a Banach algebra onto a dense subset of a strongly semi-simple Banach algebra is continuous. Thus a strongly semi-simple Banach algebra is "almost" absolute. The question arose: Is a (noncommutative) semi-simple or strongly semi-simple Banach algebra necessarily absolute? A negative answer is furnished in the present note. It is shown that in order for a Banach algebra to be absolute it is sufficient that it be strictly semi-simple and necessary that it have zero as its only nilpotent element. The latter condition is shown to be sufficient for some special Banach algebras to be absolute.

2. Necessary condition for a Banach algebra to be absolute.

THEOREM 1. *An absolute Banach algebra has no nonzero nilpotent elements.*

Proof. Suppose the Banach algebra B contains a nonzero nilpotent element. Then there exists a nonzero $v \in B$ such that $v^2 = 0$. Let A be an infinite dimensional Banach algebra such that $A^2 = (0)$. Since A is an infinite dimensional complete vector space, there exists a discontinuous linear functional on A; denote it by $f(x)$. Let $\pi(x) = f(x)v$. Since $f(x)$ is linear and $v^2 = 0$, π is seen to be a homomorphism of A into B.

Let $\|y\|$ be a Banach norm for B. Then $\|\pi(x)\| = |f(x)|\ \|v\|$ since $f(x)$ is a scalar. Since $f(x)$ is discontinuous $|f(x)|$ is not bounded and

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