A MINIMAL BOUNDARY FOR FUNCTION ALGEBRAS

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1. Introduction. An algebra $\mathfrak{A}$ of continuous functions on a compact Hausdorff space C will be understood to be a set of complex-valued functions on C which is closed under the operations of addition, multiplication, and multiplication by complex numbers. The algebra $\mathfrak{A}$ is called separating if to any two distinct points of C there exists a function in $\mathfrak{A}$ which takes distinct values at the given points. The norm $\|f\|$ of a continuous function f on a compact space is defined to be the maximum absolute value of the function. The algebra $\mathfrak{A}$ is thus a normed algebra. $\mathfrak{A}$ is called a Banach algebra if it is complete with respect to its norm, i.e., if the limit of every uniformly convergent sequence of elements of $\mathfrak{A}$ is in $\mathfrak{A}$.

An important theorem of Šilov (see [5], p. 80) asserts that if $\mathfrak{A}$ is a separating algebra of continuous functions on a compact Hausdorff space C then there is a smallest closed subset S of C having the property that every function of $\mathfrak{A}$ attains its maximum absolute value at some point of S. This set is called the Šilov boundary of $\mathfrak{A}$. A simple example is obtained by taking C to be a compact subset of the complex plane and $\mathfrak{A}$ to be the set of all continuous functions on C which are analytic at interior points; in this case the Šilov boundary of $\mathfrak{A}$ coincides with the topological boundary of C.

Given a separating normed algebra $\mathfrak{A}$ of continuous functions on a compact space C, it seems natural to ask, in view of Šilov's theorem, whether there exists a smallest subset M (not necessarily closed) of C having the property that every function in $\mathfrak{A}$ attains its maximum absolute value at some point of M. The answer in general is no. However, it will be shown (Theorem 1 below) that such a set M, called the minimal boundary of $\mathfrak{A}$, always exists if in addition it is assumed that $\mathfrak{A}$ is a Banach algebra and that there is a countable basis for the open sets of C, i.e., that C is metrizable. An example will be given to show that the metrizability of C is necessary.

If the minimal boundary M exists, it is clear that the closure of M is the Šilov boundary. An example will be given to show that M need not be closed, so that M in general is smaller than the Šilov boundary. This raises the question of the topological structure of M, which is answered (Theorem 2) by showing that M is a G_δ, i.e., a countable intersection of open sets.

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