INTRINSIC OPERATORS IN THREE-SPACE

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1. Introduction. In Euclidean three-space there are three important classical intrinsic operators, namely the intrinsic curl, the intrinsic divergence, and the intrinsic (or generalized) Laplacian. Usually they are given in terms of differential operators, but the occasion arises sometimes when they cannot be so defined. In particular if u is the Newtonian potential due to a continuous distribution, then in general u is only a function in class C^1, and consequently the usual Laplacian of u, the usual curl of grad u, and the usual divergence of grad u cannot be defined. Nevertheless, as it is easy to show, the intrinsic curl of grad u is equal to zero, the intrinsic (or generalized) Laplacian of u equals the intrinsic divergence of grad u, and furthermore Poisson's equation holds. The question arises whether the converse is true. The answer to questions of this nature is the subject matter of this paper. In particular we shall establish the following result (with the precise definitions given in the next section):

THEOREM 1. *Let D be a domain in Euclidean three-space and let v be a continuous vector field defined in D. Then a necessary and sufficient condition that v be locally in D the gradient of a potential of a distribution with continuous density is that the intrinsic curl of v be zero in D and the intrinsic divergence of v be continuous in D.*

2. Definitions and notation. We shall use the following vectorial notation: $x = (x_1, x_2, x_3)$, $\alpha x + \beta y = (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3)$, $(x, y) =$ the usual scalar product, $x \times y =$ the usual cross product, and $|x| = (x, x)^{1/2}$.

Let $v(x) = [v_1(x), v_2(x), v_3(x)]$ be a continuous vector field defined in the neighborhood of the point x_0. Then we define the upper intrinsic curl of v at x_0 to be the vector, curl $^*v(x_0) = [w_1^*(x_0), w_2^*(x_0), w_3^*(x_0)]$ where $w_j^*(x_0) = \limsup_{r\to 0}(\pi r^2)^{-1}\int_{C_j(x_0,r)}(v, dx)$, $j = 1, 2, 3$, with $C_j(x_0, r)$ the circumference of the circle of radius r and center x_0 in the plane through x_0 normal to the x_j-axis where $C_j(x_0, r)$ is oriented in the counterclockwise direction when seen from the side in which the x_j-axis points. In a similar manner using lim inf, we define the lower intrinsic curl of v at x_0, $\text{curl}_* v(x_0)$. If $\text{curl}^* v(x_0) = \text{curl}_* v(x_0)$ is finite, we call this

Received February 9, 1959. This research was supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command, under contract No. AF49 (638)-253. Reproduction in whole or in part is permitted for any purpose of the United States Government.