VIBRATION OF A NONHOMOGENEOUS MEMBRANE

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1. Introduction. We consider a simply connected two dimensional domain D with a nonhomogeneous membrane M stretched across D and fixed at the boundary Γ. Let $p(x, y) \geq 0$ be the density function of the membrane. We shall be concerned with the first eigenvalue λ_0 of the equation

$$u_{xx} + u_{yy} + \lambda p(x, y)u = 0 \tag{1}$$

subject to the condition $u = 0$ on Γ. Let K be the circle with boundary C on which a homogeneous membrane M_1 of the same mass as M is stretched. Let λ_1 be the first eigenvalue of

$$v_{xx} + v_{yy} + \lambda v = 0 \tag{2}$$

with $v = 0$ on C. In a recent paper Nehari [1] established the following interesting result.

THEOREM. (Nehari) *If* $\log p(x,y)$ *is subharmonic then*

$$\lambda_0 \geq \lambda_1 . \tag{3}$$

Nehari further showed that relaxation to the condition that $p(x, y)$ be subharmonic is not possible. In fact for the case that D is a circle and $p(x, y)$ is superharmonic the inequality in (3) is shown to be reversed.

It is the purpose of this paper to establish comparison theorems for the first eigenvalue of homogeneous and nonhomogeneous membranes *of the same shape*. That is, we shall consider the first eigenvalue of equations (1) and (2) in the same domain D subject to the boundary condition $u = 0$ and $v = 0$ on Γ respectively. We denote the first eigenvalue of the latter problem by μ and consider comparisons between λ_0 and μ. We of course have the completely trivial comparison

$$\lambda_0 \geq \mu$$

if $0 \leq p(x, y) \leq 1$ throughout D. Nehari's result pertained to the case where $p(x, y)$ had average value 1 and thus we wish to obtain relations between λ_0 and μ for density functions which may become large.

A general technique for obtaining lower bounds for the first eigenvalue for a homogeneous membrane in a domain D follows from the

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