AN ALGEBRAIC CRITERION FOR IMMERSION

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Let R be the curvature tensor of a simply connected d-dimensional ($d \geq 4$) Riemannian manifold M. T. Y. Thomas [2] has proved that if the rank of R is not too small, there exist conditions expressed in terms of polynomials in the coordinates of R which are satisfied if and only if M can be immersed in the Euclidean space R^{d+1}. The proof is existential; the polynomials are not all given explicitly. Using the notion of Grassmann algebra we shall find a single, rather simple condition on R necessary and sufficient for the existence of an immersion $i: M \rightarrow \bar{M}(K)$ with second fundamental form of rank at least four, where $\bar{M}(K)$ is a complete $(d+1)$-dimensional Riemannian manifold of constant curvature K. If coordinates are introduced this condition can be expressed algebraically in terms of polynomial equations and inequalities in the coordinates of R. The case $K=0$ yields an explicit variant of Thomas' result.

1. A differential criterion for immersion. Following [1] we fix the following notation for the structural elements associated with a d-dimensional C^∞ Riemannian manifold M: $F(M)$, the bundle of frames on M: R_a, right-multiplication of $F(M)$ by $a \in O(d)$, the group of $d \times d$ orthogonal matrices; φ, the 1-form of the Riemannian connection. Thus $\varphi = (\varphi_{ij})$ is a vertical equivariant 1-form on $F(M)$ with values in the Lie algebra of $d \times d$ skew-symmetric matrices. (We assume throughout that $1 \leq i, j, k \leq d$.) Let $\omega = (\omega_i)$ be the usual horizontal equivariant R^d-valued 1-form on $F(M)$ defined by $\omega_i(x) = \langle d\pi(x), f_i \rangle$, where x is in the tangent space $F(M)_f$ to $F(M)$ at $f = (f_1, \cdots, f_d)$ and π is the natural projection. The curvature form $\Phi = (\Phi_{ij})$ is by definition $D\varphi$, the horizontal part of $d\varphi$. In the case of 1-forms or 1-vectors we write xy, rather than $x \wedge y$, for the Grassmann product.

THEOREM 1. *Let M be a simply connected d-dimensional Riemannian manifold, $\bar{M}$ a complete $(d+1)$-dimensional Riemannian manifold of constant curvature K. Then M can be immersed in $\bar{M}$ if and only if there exists a horizontal equivariant R^d-valued 1-form $\sigma = (\sigma_i)$ on $F(M)$ such that*

$$(1) \qquad \begin{cases} \Sigma_k \sigma_k \omega_k = 0 & \\ \Phi_{ij} = \sigma_i \sigma_j + K\omega_i\omega_j & \text{(Gauss equation)} \\ D\sigma_i = 0 & \text{(Codazzi equation)}. \end{cases}$$

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