ON A CRITERION FOR THE WEAKNESS OF AN IDEAL BOUNDARY COMPONENT

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1. Exhaustion. Let F be an open Riemann surface. An *exhaustion* $\{F_n\}$ of F is an increasing (i.e., $\overline{F}_n \subset F_{n+1}$) sequence of subregions with compact closures such that $\bigcup_{n=1}^{\infty} F_n = F$. We assume that ∂F_n consists of a finite number of closed analytic curves and that each component of $F - F_n$ is noncompact. This is the most common definition used in the theory of open Riemann surfaces. Sometimes, however, we shall add the restriction that each component of ∂F_n is a dividing cycle; if this is the case we shall call the exhaustion *canonical*.

2. Weak boundary component. Let γ be an ideal boundary component of F, and let $\{F_n\}$ be a canonical exhaustion of F. Then there exists a component γ_n of ∂F_n which separates γ from F_n. Let n_0 be a fixed number and consider the component G_n of $\overline{F}_n - F_{n_0}$ $(n > n_0)$ such that $\gamma_n \subset \partial G_n$. There exists a harmonic function $s_n(p)$ on $\overline{G}_n$ which satisfies the following conditions:

(i) $s_n = 0$ on γ_{n_0} and $\int_{\gamma_{n_0}} *ds_n = 2\pi$, $(\gamma_{n_0} = \partial F_{n_0} \cap \partial G_n)$

(ii) $s_n = \log r_n = \text{const.}$ on γ_n,

(iii) $s_n = \text{const.}$ on each component $\beta_{n\nu}$ of $\partial G_n - \gamma_n - \gamma_{n_0}$ and $\int_{\beta_{n\nu}} *ds_n = 0$.

The condition $\lim_{n\to\infty} r_n = \infty$ depends neither on n_0 nor on the exhaustion. If it is satisfied, γ is said to be *weak*.

Weak boundary components were introduced for plane regions by Grötzch [1] in connection with the so-called Kreisnormierungsproblem. He called them vollkommen punktförmig. They were generalized for open Riemann surfaces by Sario [6] and discussed also by Savage [7] and Jurchescu [2]. The above definition was given by Jurchescu [2].

A noncompact subregion N whose relative boundary ∂N consists of a finite number of closed analytic curves is called a *neighborhood of* γ if γ is an ideal boundary component of N as well. Let $\{c\}$ be the family of all cycles c (i.e., unions of finite numbers of closed curves) which are in N and separate γ from ∂N. Jurchescu [2] showed that $\lambda\{c\} = 0$ if and only if γ is weak, where $\lambda\{c\}$ is the extremal length of the family $\{c\}$.

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