TRANSFORMATIONS ON TENSOR PRODUCT SPACES

MARVIN MARCUS AND B. N. MOYLS

1. Introduction. Let U and V be m- and n-dimensional vector spaces over an algebraically closed field F of characteristic 0. Then $U \otimes V$, the tensor product of U and V, is the dual space of the space of all bilinear functionals mapping the cartesian product of U and V into F. If $x \in U$, $y \in V$ and w is a bilinear functional, then $x \otimes y$ is defined by: $x \otimes y(w) = w(x, y)$. If $e_1, \cdots, e_m$ and $f_1, \cdots, f_n$ are bases for U and V, respectively, then the $e_i \otimes f_j$, $i = 1, \cdots, m$, $j = 1, \cdots, n$, form a basis for $U \otimes V$.

Let $M_{m,n}$ denote the vector space of $m \times n$ matrices over F. Then $U \otimes V$ is isomorphic to $M_{m,n}$ under the mapping ψ where $\psi(e_i \otimes f_j) = E_{ij}$, and E_{ij} is the matrix with 1 in the (i, j) position and 0 elsewhere. An element $z \in U \otimes V$ is said to be of rank k if $z = \sum_{i=1}^{k} x_i \otimes y_i$, where $x_1, \cdots, x_k$ are linearly independent and so are $y_1, \cdots, y_k$. If $R_k = \{z \in U \otimes V \mid \operatorname{rank}(z) = k\}$, then $\psi(R_k)$ is the set of matrices of rank k, in $M_{m,n}$. In view of the isomorphism any linear map T of $U \otimes V$ into itself can be considered as a linear map of $M_{m,n}$ into itself.

In [2] and [3], Hua and Jacob obtained the structure of any mapping T that preserves the rank of every matrix in $M_{m,n}$ and whose inverse exists and has this property (coherence invariance). (In [3] F is replaced by a division ring, and T is shown to be semi-linear by appealing to the fundamental theorem of projective geometry.) In [4] we obtained the structure of T when $m = n$, T is linear and T preserves rank 1, 2 and n. Specifically, there exist non-singular matrices M and N such that $T(A) = MAN$ for all $A \in M_{nn}$, or $T(A) = MA'N$ for all A, where A' designates the transpose of A. Frobenius (cf. [1], p. 249) obtained this result when T is a a linear map which preserves the determinant of every A. In [5] it was shown that this result can be obtained by requiring only that T be linear and preserve rank n. In the present paper we show that rank 1 suffices (Theorem 1), or rank 2 with the side condition that T maps no matrix of rank 4 or less into 0 (Theorem 2). Thus our hypothesis will be that T is linear and $T(R_1) \subseteq R_1$. We remark that T may be singular and still its kernel may have a zero intersection with R_1; e.g., take $U = V$ and $T(x \otimes y) = x \otimes y + y \otimes x$.

2. Rank one preservers. Throughout this section T will be a linear transformation (l.t.) of $U \otimes V$ into $U \otimes V$ such that $T(R_1) \subseteq R_1$. Here

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