MEASURES ON BOOLEAN ALGEBRAS

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This paper is concerned with the general problem of the existence of measures on Boolean algebras. A *measure* on a Boolean algebra $\mathscr{A}$ is a finitely additive, non-negative function on $\mathscr{A}$ which assumes the value one at the unit element of the algebra $\mathscr{A}$. It is known that measures on Boolean algebras always exist, and in some profusion (see, for example, [2]). We are concerned primarily with the existence of measures which are *strictly positive*; that is measures which vanish only at the zero element of the algebra. Not all Boolean algebras possess strictly positive measures, and workable necessary and sufficient conditions for the existence of a strictly positive measure have not been given. We shall give such conditions. Our results seem to represent definite progress on the general problem, although the relationship between our conditions and various conjectures is not clear. In particular, I do not know whether there is necessarily a strictly positive measure on an algebra $\mathscr{A}$ which satisfies the condition: $\mathscr{A} - \{0\}$ is the union of a countable family $\{\mathscr{A}_n\}$, such that each disjoint subclass of the class $\mathscr{A}_n$ contains at most n members. Tarski has conjectured that this is the case.

In the first section we define, combinatorially, for each subset $\mathscr{B}$ of a Boolean algebra $\mathscr{A}$ a number, $I(\mathscr{B})$, called the intersection number of $\mathscr{B}$. It is then showed that there is a strictly positive measure on $\mathscr{A}$ if and only if $\mathscr{A} - \{0\}$ is the union of a countable number of sets, each of which has positive intersection number. The intersection number is also evaluated precisely in terms of measures on $\mathscr{A}$; $I(\mathscr{B})$ is the maximum, for all measures m on $\mathscr{A}$, of inf $\{m(B) : B \in \mathscr{B}\}$. A dualized formulation of these results in terms of coverings is obtained.

The second section is concerned with the existence of countably additive measures. Necessary and sufficient conditions for the existence of such measures have been given by Maharam [3], but these conditions are not entirely satisfactory. The contribution to the problem made here is simply this: an algebra supports a countably additive strictly positive measure if and only if it has a strictly positive measure and is weakly countably distributive. (See the second section for definitions). The condition of weak countable distributivity appears thus as a very natural requirement which enables one to derive countably additive measures from finitely additive ones; the fundamental difficulties lie in the finitely additive case.

It has been shown that some of the natural conjectures on the

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