COINCIDENCE PROBABILITIES

SAMUEL KARLIN AND JAMES MCGREGOR

1. Introduction. It was shown in [14] that if $P(t) = (P_{ij}(t))$ is the transition probability matrix of a birth and death process, then the determinants

$$P\left(t;\begin{matrix} i_1 & \cdots & i_n \\ j_1 & \cdots & j_n \end{matrix}\right) = \begin{vmatrix} P_{i_1 j_1}(t) & \cdots & P_{i_1 j_n}(t) \\ \vdots & & \vdots \\ P_{i_n j_1}(t) & \cdots & P_{i_n j_n}(t) \end{vmatrix} \tag{1}$$

where $i_1 < i_2 < \cdots < i_n$ and $j_1 < j_2 < \cdots < j_n$ are strictly positive when $t > 0$. In this paper it is shown that these determinants have an interesting probabilistic significance.

(A) *Suppose that n labelled particles start out in states $i_1, \cdots, i_n$ and execute the process simultaneously and independently. Then the determinant* (1) *is equal to the probability that at time t the particles will be found in states $j_1, \cdots, j_n$ respectively without any two of them ever having been coincident* (simultaneously in the same state) *in the intervening time.*

From this statement it follows that the determinant is non-negative, and as will be seen strict positivity can be deduced from natural hypotheses, for example if $P_{i_\alpha j_\alpha}(t) > 0$ for $\alpha = 1, \cdots, n$ and every $t > 0$.

The truth of the above statement rests chiefly on the facts that the process is *one-dimensional*—its state space is linearly ordered, and that the path functions of the process are *everywhere "continuous"*. Of course the path functions are discontinuous in the ordinary sense but the discontinuities are only of magnitude one. Thus when a transition occurs the diffusing particle moves from a given state only into one of the two neighboring states, and even if the particle goes off to infinity in a finite time it either remains there or else it returns in a continuous way and does not suddenly reappear in one of the finite states. These two properties of one-dimensionality and "continuity" have the effect that when several particles execute the process simultaneously and independently, a change in the order of the particles cannot occur unless a coincidence first takes place. (The states are all stable so that with probability one a transition involves only one of the particles.)

It is also important for our results that the processes involved have the strong Markoff property of Hunt [10], [11], (see also [19]). However it is a consequence of theorems of Chung [3] that any continuous time

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