PROJECTIVE INJECTIVE MODULES

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1. Introduction. In this paper we prove several theorems about rings having a generous supply of projective injective modules. This is a curious class of rings. For instance, every module over a semisimple ring with minimum condition is both projective and injective, while over the integers only the zero module has this property. On the other hand, for some non-semisimple rings, Quasi Frobenius rings [5], every projective module is injective. For others no non-trivial projective module is injective (for example, a primary algebra over a field with radical square zero and having vector space dimension greater than two).

We begin our study in § 2 by considering primitive rings. We give (Theorem 2.1) a necessary and sufficient condition for a primitive ring to have a faithful projective injective irreducible module. By means of this condition we prove a structure theorem (Corollary 2.3) for rings having both a left and a right injective projective irreducible module with the same anihilator.

In § 3 we generalize both halves of a theorem originally proved by Thrall for finite dimensional algebras [10, Theorem 5]. This theorem states that a necessary and sufficient condition for the minimal injective [3] of the ring to be projective is that the ring have a faithful injective module which is a direct summand of every faithful module. We prove this theorem in one direction for semi-primary rings and, in the other direction, for rings with the ascending chain condition. It should be noted that we have rephrased the theorem to eliminate the duality given by the field. We find that this can be replaced by the dual concepts, projective and injective.

Throughout the paper we shall only consider rings with identity 1 and modules over such rings on which 1 acts like identity. " Minimum condition " means minimum condition on left ideas [1].

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2. Projective injective irreducibles. We shall begin by considering primitive rings. Recall that a (right) primitive ring R has a faithful irreducible right module M [7, p. 4]. The module M is always the homomorphic image of R, and if M is projective then M is induced by

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