THE H_p-PROBLEM AND THE STRUCTURE OF H_p-GROUPS

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1. Introduction. Let G be a group, p a prime, and $H_p(G)$ the subgroup of G generated by the elements of G which do not have order p. In a research problem in the Bulletin of the American Mathematical Society, one of the authors posed the following problem: is it always true that $H_p(G) = 1$, $H_p(G) = G$, or $[G : H_p(G)] = p$? This problem is easily settled in the affirmative for $p = 2$, and a similar answer was recently given for $p = 3$ ([5]). In this paper (Section 2) we give an affirmative answer for the case that G is finite and not a p-group. Furthermore (Section 3) we are able to give a rather precise description of the structure of G in the most interesting case, when $[G : H_p(G)] = p$. This structure theorem depends heavily on the deep results of Hall and Higman ([4]) and Thompson ([6]) on finite groups. If $H\ (\neq 1)$ is a finite group and there exists a group G such that $H_p(G)$ is isomorphic to H, where $H_p(G) \neq G$, then we call H an H_p-group; it is seen that H_p-groups are natural generalizations of "Frobenius groups." By a Frobenius group we mean a finite group G possessing an automorphism σ of prime order p such that $x^\sigma = x$ if and only if $x = 1$. It is easy to show that this implies

$$x^{1+\sigma+\cdots+\sigma^{p-1}} = x(x^\sigma)\cdots(x^{\sigma^{p-1}}) = 1 ,$$

for all x in G. This last equation characterizes H_p-groups,[1] and as a generalization of Thompson's result ([6]) that Frobenius groups are nilpotent, we show that H_p-groups are solvable, among other things.

Throughout the paper, if B is a group, A a subgroup of B, then $N_B(A)$ and $C_B(A)$ mean, respectively, the normalizer and centralizer of A in B. By $Z(A)$ we mean the center of A.

2. The H_p-problem. Let G be a group, and let $H = H_p(G)$. Suppose

(1) G is finite,

(2) G is not a p-group,

(3) the index of H in G is greater than p,

(4) G is a group of minimal order satisfying (1), (2), (3). Note that every element of G which is not in H has order p.

Let q be a prime dividing $[G : 1]$, $q \neq p$, and let Q be a Sylow q-group of G; then Q is also a Sylow q-group of H. Let $N = N_G(Q)$; then

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[1] Unless the group is a p-group; see Theorem 2.