THE RING OF NUMBER-THEORETIC FUNCTIONS

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Introduction. The set Ω of all functions $\alpha(n)$ on $N = \{1, 2, 3, \cdots\}$ to the complex field F forms a domain of integrity under ordinary addition, and *arithmetic product* defined by: $(\alpha \cdot \beta)(n) = \sum \alpha(d)\beta(n/d)$, summed over all $d \mid n$, $d \in N$. The group of units of this domain contains as a subgroup the set of all multiplicative functions. Against this background, the "inversion theorems" of number theory appear as obvious consequences of ring operations, and generalizations of the standard functions arise in a natural way. The domain Ω is isomorphic to the domain P of formal power series over F in a countable set of indeterminates. The latter part of the paper is devoted to proving that the theorem on unique factorization into primes, up to order and units, holds in P and hence in Ω.

1. **Definition.** The class Ω of all number-theoretic functions α, [4; Ch. IV], i.e., functions $\alpha(n)$ on the set N of natural numbers $n = 1, 2, 3 \cdots$ to the complex field F, forms a domain of integrity (commutative, associative ring with identity and no proper divisors of zero) under ordinary addition: $(\alpha + \beta)(n) \equiv \alpha(n) + \beta(n)$, and an operation, frequently occurring in number theory in various disguises, which we call the arithmetic product:

$$(\alpha \cdot \beta)(n) \equiv \sum \alpha(d)\beta(d')$$

the summation extending over all ordered pairs (d, d') of natural numbers such that $dd' = n$.

The commutativity $\alpha \cdot \beta = \beta \cdot \alpha$ follows from the fact that the correspondence $(d, d') \to (d', d)$ is one-to-one on such a set of ordered pairs to (all of) itself, while the associative law $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$ can be verified by observing that, in either association, $(\alpha \cdot \beta \cdot \gamma)(n) = \sum \alpha(d)\beta(d')\gamma(d'')$, summed over all ordered triples (d, d', d'') with $dd'd'' = n$.

The zero 0 and additive inverse $-\alpha$ of α are of course the functions defined by $0(n) \equiv 0$, and $(-\alpha)(n) \equiv -\alpha(n)$, and one sees at once that the function ε with $\varepsilon(1) = 1$, $\varepsilon(n) = 0$ for $n > 1$, is the identity: $\varepsilon \cdot \alpha = \alpha$ for all α of Ω.

That the ring Ω has no proper divisors of zero may be seen in various ways, three of which occur incidentally in the following sections (2, 4, 5).

2. **A norm for number-theoretic functions.** A function $N(\alpha)$ on

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