ON DERIVATIONS IN DIVISION RINGS

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We are concerned with studying division rings in which Lie rings of derivations are acting. The results include the determination of dimension over the constant subring, an outer Galois theory, and miscellaneous results on inner automorphisms and powers of derivations.

Let A be a ring with an identity 1 and B be a subring of A containing 1.

1. The mappings $R_x : y \to yx$, $L_x : y \to xy$, $I_x : y \to x^{-1}yx$, $x \in A$ are called right multiplications, left multiplications and inner automorphisms respectively. For any subset N of A, $R_N = \{R_x \mid x \in N\}$, $L_N = \{L_x \mid x \in N\}$, $I_x = \{I_x \mid x \in N\}$. D is a derivation of B into A if and only if $(x + y)D = xD + yD$ and $(xy)D = xDy + xyD$ for all $x, y \in A$. The set of all such mappings is denoted by $\mathrm{Der}(B, A)$. If $B = A$, D is called a derivation in A and the set of these denoted by $\mathrm{Der}(A)$. If $D_1, \cdots, D_s \in \mathrm{Der}(A)$, we have for all $x \in A$

$$(1) \qquad R_x D_1^{k_1} \cdots D_s^{k_s} = \sum \left\{ \binom{k_1}{i_1} \cdots \binom{k_s}{s_s} D_1^{k_1 - i_1} \cdots D_s^{k_s - i_s} R_{xD_1^{i} \cdots D_s^{i_s}} \mid 0 \le i_j \le k_j,\ j = 1, \cdots, s \right\}.$$

For all $D, D' \in \mathrm{Der}(A)$, $[DD'] = DD' - D'D \in \mathrm{Der}(A)$ and, if A has prime characteristic p, $D^p \in \mathrm{Der}(A)$. $\{x \mid xy - yx = 0 \text{ for all } y \in B;\ x \in A\}$ is called the centralizer of B in A. If c belongs to the centralizer of B in A, $DR_c \in \mathrm{Der}(B, A)$. The centralizer of A in A is called the center of A. Let C be the center of A. $\mathscr{D}$ is a Lie ring (Lie ring over C) of derivations in A if and only if $\mathscr{D} \subseteq \mathrm{Der}(A)$ and for all $D, D' \in \mathscr{D}$, $D - D' \in \mathscr{D}$, $[DD'] \in \mathscr{D}$ $(DR_c \in \mathscr{D})$. If A has prime characteristic p, $\mathscr{D}$ is restricted if, in addition, $D^p \in \mathscr{D}$.

For $x \in A$, the mapping $I'_x : y \to yx - xy$ is a derivation called an inner derivation. For $N \subseteq A$, $I'_N = \{I'_x \mid x \in N\}$. The elements of $\mathrm{Der}(A)$ not in I'_A are called outer derivations. Lie ideals are defined in the usual way for Lie rings, restricted or not, over C or not. The inner derivations in $\mathscr{D}$ form a Lie ideal in $\mathscr{D}$.

Let T be a subset of $\mathrm{Der}(B, A)$. The set of $x \in B$ such that $xD = 0$ for all $D \in T$ is a subring of B which we call the subring of T-constants and which we denote by $B(T)$. If $x \in B(T)$ and x has a multiplicative inverse x^{-1} in B, then $x^{-1} \in B(T)$. The set of derivations D in A such

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