STATISTICAL METRIC SPACES

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Introduction. The concept of an abstract metric space, introduced by M. Fréchet in 1906 [2], furnishes the common idealization of a large number of mathematical, physical and other scientific constructs in which the notion of a "distance" appears. The objects under consideration may be most varied. They may be points, functions, sets, and even the subjective experiences of sensations. What matters is the possibility of associating a non-negative real number with each ordered pair of elements of a certain set, and that the numbers associated with pairs and triples of such elements satisfy certain conditions. However, in numerous instances in which the theory of metric spaces is applied, this very association of a *single* number with a pair of elements is, realistically speaking, an over-idealization. This is so even in the measurement of an ordinary length, where the number given as the distance between two points is often not the result of a single measurement, but the average of a series of measurements. Indeed, in this and many similar situations, it is appropriate to look upon the distance concept as a statistical rather than a determinate one. More precisely, instead of associating a number—the distance $d(p, q)$—with every pair of elements p, q, one should associate a distribution function F_{pq} and, for any positive number x, interpret $F_{pq}(x)$ as the probability that the distance from p to q be less than x. When this is done one obtains a generalization of the concept of a metric space—a generalization which was first introduced by K. Menger in 1942 [5] and, following him, is called a statistical metric space.

The history of statistical metric spaces is brief. In the original paper, Menger gave postulates for the distribution functions F_{pq}. These included a generalized triangle inequality. In addition, he constructed a theory of betweeness and indicated possible fields of application.

In 1943, shortly after the appearance of Menger's paper, A. Wald published a paper [14] in which he criticized Menger's generalized triangle inequality and proposed an alternative one. On the basis of this new inequality Wald constructed a theory of betweeness having certain advantages over Menger's theory [15].

In 1951 Menger continued his study of statistical metric spaces in a paper [7] devoted to a resume of the earlier work, the construction of several specific examples and further considerations of the possible applications of the theory. In this paper Menger adopted Wald's version

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