AN INVERSION THEOREM FOR LAPLACE-STIELTJES TRANSFORMS

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E. Phragmén [2; p. 360] showed that under certain assumptions of boundedness for $F(x)$,

$$\lim_{s\to+\infty}\int_0^t F(\tau)[1-\exp(-e^{(t-\tau)s})]\,d\tau=\int_0^t F(\tau)d\tau\ .$$

If we write $1-\exp(-e^{s(t-\tau)})=\sum_1^\infty(-1)^{n+1}e^{nx(t-\tau)}/n!$ in the above formula, and interchange sum and integral, we formally obtain

$$\lim_{s\to\infty}\sum_1^\infty\frac{(-1)^{n+1}}{n!}e^{nst}\int_0^t e^{-ns\tau}F(\tau)d\tau=\int_0^t F(\tau)d\tau\ .$$

G. Doetsch [1; pp. 286–288] showed that for reals s, if $f(s)=\int_0^\infty e^{-s\tau}F(\tau)d\tau$ converges absolutely in some half-plane, then

$$\int_0^t F(\tau)d\tau=\lim_{s\to+\infty}\sum_1^\infty\frac{(-1)^{n+1}}{n!}f(ns)e^{nst}\ \text{for}\ t>0\ .$$

This paper will generalize this result to Laplace-Stieltjes transforms

$$\text{(I)}\qquad f(s)=\int_0^\infty e^{-st}\,d\alpha(t)$$

and will eliminate the assumption of absolute convergence. Unless specifically written otherwise, all integrals will be evaluated from 0 to $+\infty$ and all summations from 1 to ∞. We shall need the following two propositions [3; pp 39,41]:

LEMMA 1. *If the integral*

$$f(s_0)=\int e^{-s_0t}\,d\alpha(t)$$

converges with $Rs_0>0$, *then*

$$f(s_0)=s_0\int e^{-s_0t}\,\alpha(t)dt-\alpha(0)$$

and $\int e^{-s_0t}\,\alpha(t)dt$ *converges absolutely if* s_0 *is replaced by any number with larger real part.*

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