ON CERTAIN SINGULAR INTEGRALS

BENJAMIN MUCKENHOUPT

1. Introduction. The purpose of this paper is to consider a modification of the Hilbert transform and the singular integrals treated by Calderon and Zygmund in [1] and [3], and to use the results to generalize some standard results on fractional integration. In the one dimensional case the Hilbert transform of a function $f(x)$ is essentially the integral $\int_{-\infty}^{\infty} \frac{f(x-t)}{t} dt$. In the one dimensional case the transform to be considered will be a convolution with $\frac{1}{|t|^{1+i\gamma}}$ instead of with $\frac{1}{t}$. Throughout this paper γ will denote a real number not zero. As in the Hilbert transform case there is trouble with the definition; for the Hilbert transform this is solved by taking a Cauchy value at the origin. The obvious extension of this method was used by Thorin [6] when he considered a transform of the type

$$\lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} \frac{f(x-t) - f(x+t)}{t^{1+i\gamma}} dt .$$

Here and subsequently ε will always be greater than 0 and the limits in ε will be one sided. In this case, however, obtaining cancellation by taking a Cauchy value is unnecessary; the kernel already has sufficient oscillations to accomplish this. The integral $\lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} \frac{f(x-t)}{t^{1+i\gamma}} dt$ will not, in general, exist, but by using some suitable summation procedure, it may be given meaning. Starting with two such methods, it is shown that this transform has the usual singular integral properties. Specifically, for functions in a Lebesgue L^p class $1 < p < \infty$, it is shown that the summation procedure converges in L^p and that the resulting transformation is bounded in L^p. For $p = 1$ substitute results are obtained. Furthermore, for functions in $L^p, 1 \leq p < \infty$, the summation procedure is shown to converge pointwise almost everywhere.

Carried along simultaneously with the preceding is the n dimensional extension of the sort considered by Calderón and Zygmund for the Hilbert transform. In Euclidean n space, E^n, let $x = (x_1, x_2 \cdots x_n)$, $|x| = (x_1^2 + \cdots x_n^2)^{\frac{1}{2}}$ and $dx = dx_1 \cdots dx_n$. The transforms to be considered are of the form

$$\int_{E^n} \frac{f(x-t)}{|t|^{n+i\gamma}} \Omega(t) \, dt .$$

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